List of Proposed Problems for August, 2013 A.I.M. Meeting on Sections of Convex Bodies

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Contents

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1 Max/min Perimeter of central cross-sections

Proposed by Hermann König

1. Find the central cross section $u^\perp := \{x \in \mathbb{R}^n : x \cdot u = 0\}$ of $B_\infty^n = \{x \in \mathbb{R}^n : |x_i| \leq 1\}$ with the largest perimeter.
2. What is the central cross section of B_∞^n with smallest perimeter?
3. What if instead of perimeter we take the mean width or intrinsic volume?
4. What about other codimensions?
5. What about other measures such as gaussian measures?
6. What about perimeter of projections?

2 Bounded Projection Inequality

Proposed by Mathieu Meyer

Let $\mathcal{K}_{os}^n$ be the collection of all origin symmetric convex bodies in $\mathbb{R}^n$. Find an asymptotic bound for the quantity

$$\max_{K \in \mathcal{K}_{os}^n} \min_{u,v \in S^{n-1}} \frac{P_{v,u^\perp}(K)}{\mathrm{vol}_{n-1}(K \cap u^\perp)}, \tag{1}$$

where $P_{v,u^\perp}(K)$ is the projection of K on $u^\perp$ in the direction $v \in S^{n-1}$.

One should note that 2^n is a rough upper bound for (1) because for every hyperplane H in $\mathbb{R}^n$, there is a projection P_H so that $\|P_H\| \leq 2$. This would imply $P_H K \subseteq 2(K \cap H)$ and hence $\|P_H K\| \leq 2^{n-1}$.

One should also note that

$$\min_{u,v \in S^{n-1}} \frac{P_{v,u^\perp}(K)}{\mathrm{vol}_{n-1}(K \cap u^\perp)} \tag{2}$$

is affine invariant. If K is an ellipsoid then the quantity in (2) is 1 so (1) is always greater than or equal to 1.

3 Preservation of Convexity Under the Intersection Body Operator

Proposed by Maria Alfonseca

Suppose K is an origin-symmetric starbody in $\mathbb{R}^n$. let $\rho_K(u) := \max\{a : au \in K\}$ and define the intersection body IK of K as the body defined by

$$\rho_{IK}(u) = \mathrm{Vol}_{n-1}(K \cap u^{\perp}) \qquad \forall u \in S^{n-1}. \tag{3}$$

Define the *convex kernel* of K as the intersection of all convex bodies M such that $m \subseteq K$. What conditions on the convex kernel of K would imply IK is convex?

4 Cotype and Volume Ration of $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$

Proposed by Carsten Schütt

Find the cotype and volume ration of $\ell_p^n \otimes_\pi \ell_q^n \otimes_\pi \ell_r^n$.

5 Banach-Mazur Distance Problem

Proposed by Mark Rudelson

Let $\mathcal{K}_{os}^n$ denote the space of origin-symmetric, convex bodies in $\mathbb{R}^n$ and let d_{BM} is the Banach-Mazur distance.

Let $\alpha \in (0,1)$ and $n \in \mathbb{N}$. Suppose $K, L \in \mathcal{K}_{os}^n$ are such that there exists $b > 0$ so

$$d_{BM}(K \cap E, L \cap E) \leq b \qquad \forall E \in \mathcal{K}_{os}^{\alpha n}. \tag{4}$$

Does there exists a function f so that

$$d_{BM}(K, L) \leq f(b, \alpha)? \tag{5}$$

It's worth noting that M. Rudelson showed during one of the problem sessions that this question is trivially true if d_{BM} is replaced with the geometric distance d_g.

6 Injective Operators which Correspond with the Intersection Body Operator

Proposed by Richard Gardner

Let $\mathcal{K}_o^n$ denote the collection of convex bodies (i.e., compact convex sets with nonempty interior) in $\mathbb{R}^n$ and let $\mathcal{K}_{os}^n$ denote the origin-symmetric members of $\mathcal{K}_o^n$. Is there an $F : \mathcal{K}_o^n \to \mathcal{K}_{os}^n$ such that F is continuous, affine invariant, injective (modulo translations and reflections in the origin), and $F|_{\mathcal{K}^n} = I$, where I is the intersection body operator.

7 Discrete Tomography

Proposed by Richard Gardner

(Discrete Aleksandrov projection theorem.) If two n-dimensional centrally symmetric convex lattice sets in $\mathbb{Z}^n$ are such that the cardinality of their projections on any lattice hyperplane is the same, are they equal up to translation? This question appears with the restriction $n \geq 3$ in [GGZ].

For $n = 2$, the answer is negative, but only an isolated counterexample is known, consisting of three non-congruent origin-symmetric convex lattice sets, each with 11 points. See Figure 4 in the above paper. It is possible that the answer is affirmative when $n = 2$ for sets containing sufficiently many points.

8 Geometric Problems on Sections of Convex Bodies

Proposed by Richard Gardner

Let $\mathscr{G}(n, k)$ denote the set of k-dimensional subspaces of $\mathbb{R}^n$, where $2 \leq k \leq n-1$ is fixed in the following problems. If K is a set let $\operatorname{int} K$ denote the interior of K. Also let o denote the origin in $\mathbb{R}^n$ and if $S \in \mathscr{G}(n, k)$, let $K|S$ denote the projection of K on S.

1. (See [G, Problem 7.4 and Note 7.2].) Suppose $o \in \operatorname{int} K$ and all sections $K \cap S$, $S \in \mathscr{G}(n, k)$, are affinely equivalent. Does it follow that K is an ellipsoid? Here K is a star body but the problem is also open when K is convex.

2. (See [G, Problem 3.3 and Note 3.2].) Same problem as in (1) but for projections, i.e. if all projections $K|S$, $S \in \mathscr{G}(n, k)$, are affinely equivalent, is K is an ellipsoid? Here K is convex.

3. (See [G, Problem 7.1 and Note 7.1].) Suppose K and L are star bodies and that $K \cap S$ is homethetic to $L \cap S$ for every $\mathscr{G}(n, k)$. Does it follow that K is homothetic to L? The answer is affirmative if K and L are convex and $o \in \operatorname{int} K \cap \operatorname{int} L$; see [G, Theorem 7.1.1].

4. (See [G, Problem 7.3 and Note 7.1].) Suppose K and L are star bodies and that $K \cap S$ is congruent to $L \cap S$ for every $\mathscr{G}(n, k)$. Does it follow that $K = \pm L$? It has been shown that the answer is affirmative for $k = 2$ when "congruent to" is replaced by "a rotation of". For more on this see [R].

5. (See [G, Problem 3.2 and Note 3.1].) Suppose K and L are convex bodies and that $K|S$ is congruent to $L|S$ for every $\mathscr{G}(n,k)$. Does it follow that K is a translate of $\pm L$? A corresponding affirmative answer for $k = 2$ and rotations is given by Ryabogin in [R]. Moreover, significant progress on the question for congruent projections has been made recently by F. Nazarov; this information was communicated by Dimtry Ryabogin.

9 Vertex Index Problems

Proposed by Alexander Litvak

1. Prove that the vertex index of the n-dimensional Euclidean ball B_2^n is $2n^{3/2}$, where the vertex index of a centrally-symmetric convex body $K = -K \subset \mathbb{R}^n$ is defined as

$$\text{vein}(K) = \inf\left\{\sum_i \|p_i\|_K \mid K \subset \text{conv}\{p_i\}\right\}.$$

2. Find the best possible upper bound for the vertex index of 3-dimensional centrally-symmetric convex body.

3. Provide an upper bound for the vertex index of a convex body $K \subset \mathbb{R}^n$

$$\text{vein}(K) = \inf\left\{\sum_i \|p_i\|_{K-a} \mid a \in K,\ K - a \subset \text{conv}\{p_i\}\right\}.$$

10 Reconstruction of Polytopes with Few Facets (or Few Vertices)

These questions are motivated by the problem of efficiently estimating polytopes with few facets or few vertices from random points.

1. (Identifiability from moments) Let the k-th moment tensor of a convex body be the k-th moment tensor of the uniform distribution on it. For any positive integer t, is there $k = k(t)$ (independent of d) so that the following map is injective: The map that takes a d-dimensional polytope with d^t facets and maps it to its first k moment tensors.

   Known:

(a) [GLPR] Can recover a d-dimensional polytope with n vertices from $O(dn)$ moments along d random directions.

(b) [FJK] Can estimate a parallelepiped efficiently from first 4 moment tensors.

(c) [AGR] Can estimate any simplex efficiently from first 3 moment tensors.

In the affirmative case, is there an efficient reconstruction algorithm? Namely, is there an algorithm running in time polynomial in n whose input are the first k moment tensors of any given polytope P as before, and that outputs a polytope within 1/10 Hausdorff distance of P? What if the tensors are not known exactly but come from a sample approximation from a sample of size polynomial in n for any fixed t?

2. (Stability of reconstruction from relative central sectional areas) Is it true that for any positive integer t there is a positive integer $t' = t'(t)$ such that for any two isotropic centrally symmetric d-dimensional polytopes P, Q with at most d^t facets we have: If for all $\theta \in S^{d-1}$

$$\left| \frac{\mathrm{Vol}_{n-1}(P \cap \theta^\perp)}{\mathrm{Vol}(P)} - \frac{\mathrm{Vol}_{n-1}(Q \cap \theta^\perp)}{\mathrm{Vol}(Q)} \right| \leq \frac{1}{d^{t'}},$$

then the Hausdorff distance between P and Q is at most 1/10?

In the affirmative case, is there an efficient reconstruction algorithm? Namely, is there an algorithm running in time polynomial in n that when given access to the value of relative central sectional areas of any given polytope P as before, to within additive error $1/d^{t'}$, it outputs a polytope within 1/10 Hausdorff distance of P?

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