Sections of convex bodies

organized by
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Workshop Summary

The study of geometric properties of convex bodies based on information about sections of these bodies has important applications to many areas of mathematics and science. A new approach to sections and projections of convex bodies, based on methods of Fourier analysis, has recently been developed. The idea is to express geometric characteristics of a body in terms of the Fourier transform and then use methods of harmonic analysis to solve geometric problems. This approach has led to several results including Fourier analytic solutions of the Busemann-Petty and Shephard problems, characterizations of intersection and projection bodies, extremal sections and projections of certain classes of bodies. The most recent results include solutions of several longstanding uniqueness problems, discovery of stability in volume comparison problems and its connection to hyperplane inequalities. Still, many most natural geometric problems on sections of convex bodies remain open.

As a result of morning lectures and afternoon discussions, the participants were able to identify many open problems and promising ways to attack them. The workshop has created new collaborations which will undoubtedly lead to new results in the area.

Questions pertaining to intersection and projection bodies were discussed in lectures by M. Meyer and J. Kim. The intersection body of a star body is defined as the star body whose radius in each direction is equal to the volume of the central section of the body perpendicular to the direction. A classical theorem of H. Busemann says that the intersection body of a symmetric convex body is convex. It is still unknown if a weaker assumption can guarantee convexity and how to find all fixed points of intersection body operator. One of the discussion groups studied the question about how much of 'convexity' can be preserved or improved under the intersection body operation.

Another group discussed the following question of M. Meyer. Let $\mathcal{K}_{os}^n$ be the collection of all origin symmetric convex bodies in $\mathbb{R}^n$. Find an asymptotic bound for the quantity

$$\max_{K\in\mathcal{K}_{os}^n}\ \min_{u,v\in S^{n-1}} \frac{|P_{v,u^\perp}(K)|}{|K\cap u^\perp|},$$

where $P_{v,u^\perp}(K)$ is the projection of K on $u^\perp$ in the direction $v\in S^{n-1}$. The quantity can be viewed as a measure of symmetry of convex bodies.

The talk of R. Gardner inspired the creation of a Discrete Tomography group. The work of the group was mainly concerned with questions on projections and sections of convex lattice sets. The discrete brightness function of a lattice set S assigns to every direction u the cardinality of the projection of the set onto the hyperplane orthogonal to u. Note that there are only finitely many values of the discrete brightness function of S which are different from the cardinality of S.

The following question can be considered as a discrete version of Aleksandrov's theorem on projections of centrally-symmetric convex bodies. Do there exist two distinct centrally-symmetric convex lattice sets in $\mathbb{Z}^n$ with the same discrete brightness function? It was shown by Gardner, Gronchi and Zong that the answer is negative for $n = 2$, but the only known counterexample consists of three centrally-symmetric convex lattice sets in $\mathbb{Z}^2$, each with 11 points. The Discrete Tomography group was working on constructing counterexamples with larger number of points or looking for obstructions for their existence.

The Discrete Tomography group was also working on the following question posed by A. Koldobsky, which relates the number of lattice points in a lattice polytope (i.e. "discrete volume") to the maximal number of lattice points in its central sections (i.e. "discrete maximal section"): Let P be an n-dimensional lattice polytope in $\mathbb{R}^n$. Is it true that the number of lattice points of P is bounded above by

$$|P|^{1/n} \max_{u \in S^{n-1}} \#(P \cap u^{\perp} \cap \mathbb{Z}^n),$$

up to a universal constant? Here $|P|$ denotes the Euclidean volume of P and $\#A$ denotes the cardinality of a set A.

H. Koenig gave a lecture on extremal problems concerning sections of convex bodies, with a special emphasis on the slicing problem and sections of the cube. There was a discussion group working on the problem of finding sections of the cube that have minimal or maximal perimeters. As a result, the participants were able to derive a Fourier analytic formula expressing the size of perimeters of the cube. The main problem was to find extremal values using the formula. Participants had a few ideas about how to compute the minimum. In dimension 3 the problem was solved earlier by Pelczynski.

M. Rudelson asked a question whether the Banach-Mazur distance between sections of convex bodies can be used to bound the distance between the bodies. More precisely, consider two origin symmetric convex bodies $K, L \subset \mathbb{R}^n$ such that there exists $b > 0$ so

$$d_{BM}(K \cap E, L \cap E) \leq b, \qquad \text{for all } E \text{ of dimension } [n/2].$$

Does there exists a function f so that

$$d_{BM}(K, L) \leq f(b)?$$

It became clear that if one of the bodies is an ellipsoid, then the answer is known to be true and follows from works of Bourgain, Mankiewicz and Tomczak-Jaegermann. The group tried to consider the case when one of the bodies is the cube or crosspolytope, as well as related questions. The group conjectured that if the above statement is true for a fixed body K and all bodies L then K must satisfy some special "good" properties.