Weighted singular integral operators and non-homogenous harmonic analysis

organized by
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Workshop Summary

This workshop, sponsored by AIM and the NSF, focused on recent developments on weighted inequalities for singular integral operators and its connection with questions in geometric measure theory and PDE. We dedicated most of the time to the following two central problems:

- The harmonic analytic question of establishing a characterization of the two-weight inequality for singular integral operators. This characterization would be a profound extension of the famous T1 Theorem of David and Journé. There exist partial solutions to the conjecture assuming extra side conditions on the weights.
- The geometric measure theoretic question of characterizing rectifiable sets E in terms of boundedness of the Riesz transforms in $L^2(E)$. This is currently one of the most important open conjectures in the interface between harmonic analysis and geometric measure theory.

We spent the mornings at AIM overviewing the leading approaches on the above mentioned questions, as well as other problems in non homogeneous harmonic analysis that were of interest to the audience. The afternoon sessions were filled with discussion on a set of open problems. Those new targets were set on the first afternoon after a brainstorming activity.

The workshop started with a survey talk on non-homogeneous harmonic analysis by A. Volberg. The central question of non-homogeneous harmonic analysis consist in characterizing L^2 boundedness of Calderón-Zygmund operators in the presence of non-homogeneous measures. The randomized martingale method developed by Nazarov, Treil and Volberg has not only provided the answer to the question, but has also inspired other works. In our second talk, we saw one of those instances. T. Hytönen spoke about recovering Calderón-Zygmund operators from Haar shifts, via a randomization procedure. His work resulted in the resolution of the A_2 conjecture.

We dedicated the second morning of the workshop to the two weight problem for the Hilbert transform. E. Sawyer and M. Lacey presented an overview of their current proof on the subject (joint with I. Uriarte-Tuero). Their characterization relies on testing the operator on functions of bounded fluctuation (one would like to restrict to testing on characteristic functions on cubes), on the other hand, they don't assume any side condition on the measures while previous proofs do.

There were 3 talks related to David-Semmes conjecture, one by X. Tolsa, another by S. Hofmann and the third one by Svitlana Mayboroda. X. Tolsa introduced the main partial results currently known in connection with the David-Semmes problem: the two-dimensional approach of Mattila, Melnikov, and Verdera resting on Menger curvature; results of David and Semmes linking boundedness of the entire class of Calderń-Zygmund operators with rectifiability of sets; and more recent advances of Tolsa himself showing that existence of principal values of Riesz transforms implies rectifiability. S. Hofmann highlighted a possible approach to the David-Semmes problem via the theory of partial differential equations, viewing the Riesz transform as a gradient of the single layer operator. In this context, he moreover presented his recent work with J. Martell and I. Uriarte-Tuero which culminated at characterization of rectifiability for a large class of sets via the A^∞ condition on harmonic measure. S. Mayboroda discussed the impetus of the David-Semmes conjecture on questions of removability of singularities of bounded analytic (in higher dimensions, Lipschitz harmonic) functions, Painlevé problem, Vitushkin conjecture. She also discussed a recent joint work of Mayboroda and Volberg on equivalence of rectifiability to finiteness of the square function associated to the Riesz transform.

The were a set of talks related to sharp weighted estimates. A talk by A. Volberg on the Bellman function approach to the proof of the A_2 Conjecture and Weak Muckenhoupt Wheeden Conjecture. And a talk by M. Reguera on a counterexample of a long-standing conjecture by Muckenhoupt-Wheeden at the end point $p = 1$. We also enjoyed a talk by I. Uriarte-Tuero on an application of non-homogeneous weighted theory in Quasiconformal Theory, resolving a conjecture due to K. Astala.

The afternoon sessions were dedicated to group discussion. During the first afternoon an "ask the expert" session and a brainstorming activity allowed us to identify an array of open questions. The rest of the afternoons the participants divided into groups to work on those questions according to their preference. We provided the open problems below.

- **Rough kernels** The main question here is concerned with finding sharp estimates for operators with rough kernels. It will be very desirable if a decomposition on averages of dyadic operators, in the spirit of T. Hytönen's work on Calderón-Zygmund operators, could be provided.
- **Two weight bump conditions and the Bellman Approach** Two weight bump conditions is a universal conditions of A_2-type that are conjectured to be sufficient for the two weight boundedness of all Calderón–Zygmund operators simultaneously. The usual A_2 assumption is known to be necessary but not sufficient even if used with a smoothing kernel with a tail (Nazarov). This is why Carlos Pérez promoted these so-called "bump" conditions, where the average is taken in the sense of Orlicz norms. His conditions are known to be sharp: the integral assumption on Orlicz gauge functions cannot be weakened. Moreover, for the maximal operator the sharp bump conditions are known to be sufficient. There was much progress recently in proving their sufficiency for particular classes of Calderón–Zygmund operators. But the proof seemed elusive for *all* such operators. The meeting was very important in allowing the experts of 3 different approaches (Lerner's approach, stopping time approach of Lacey–Petermichl–Reguera, and Bellman approach) to talk extensively to each other and to cut the dead-end approaches. As a result the problem was solved 2 days after the conference by efforts of Alexander Reznikov (the graduate student at Michigan Sate University) and Alexander Volberg, David Cruz-Uribe, Fedor Nazarov, Sergei Treil and Michael Lacey. Moreover, there are two solutions now: one due to David Cruz-Uribe, Michael Lacey, Alexander Reznikov and Alexander Volberg uses stopping time and Lerner's ideas; another due to Alexander Reznikov, Alexander Volberg, Fedor Nazarov, Sergei Treil uses the Bellman function technique.

- **Sharp weighted estimates in multi-parameter setting** Finding sharp weighted estimates in the multi-parameter setting is a hard task. The question for the rectangular maximal function is still open. The group was discussing possible extensions of the competing proofs in the one parameter setting, unfortunately they don't seem to provide any insight onto this one.
- **Sharp estimates using dyadic shifts in non-weighted setting** Any reasonable estimate on BMO using dyadic shifts decomposition is an interesting question. Of course BMO has many equivalent norms, but the Bellman function approach leads to sharp relationship between them. This was discussed with the applications in mind of sharp norm estimates of classical singular integrals on BMO space. Classical John–Nirenberg's distribution function estimate, where the measure of the set of points on which a BMO function belongs to a *arbitrary finite set* of intervals can be generalized, and found in a sharp way.
- **David-Semmes Conjecture** The group has concentrated on two questions. The first one is the profound connections of the the David-Semmes problem with the PDE's, its possible equivalent formulations via the local estimates on the harmonic Green function and the square function estimates on the second gradient of the single layer, as well as the role of the maximum principle suggested by the recent work of Eiderman, Nazarov, Volberg. The second question actively pursued was the particular properties of the kernel of the Riesz transform which indicate that its boundedness alone would imply rectifiability. Connected to this is the question for what other singular operators, even in dimension two, such an implication could be valid. These issues are naturally linked with equivalent capacities, some of which bring us back to PDEs.
- **Composition of paraproducts** When one consider dyadic paraproducts, boundedness is determined by a BMO or L^∞ condition, depending on the nature of the paraproduct. When considering the composition of such two, one could have boundedness even if one or both of them are unbounded. We would like to find characterizations for boundedness of such compositions in the spirit of the $T1$ theorem of David and Journé. Although some instances have already been understood, the difficult cases are still open.
- A_p **in non-homogeneous setting** The question of interest here is to find necessary and sufficient conditions for boundedness of Calderón-Zygmund operators in the A_2 setting when the measure accompanying the weight w is not Lebesgue but a general measure. C. Pérez and J. Orobitg provided a condition, which is the analogous of the A_p condition in the Lebesgue setting, unfortunately this condition is sufficient but not necessary. The group was discussing the possibility of considering smaller collection of intervals to test the A_p condition and what would be the properties of such a collection.