Numerical invariants of singularities and higher-dimensional algebraic varieties

organized by
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Workshop Summary

Our workshop took place at the American Mathematical Institute from July 31 to August 4. The organizers of the workshop were N. Budur, L. Ein, R. Lazarsfled, M. Mustaţă and V. Shokurov There were about thirty participants, mathematicians working on related areas such as birational algebraic geometry, singularities theory, complex differential algebraic geometry and commutative algebra.

In recent years, certain numerical measures of the singularities of a divisor or holomorphic function have appeared in a surprisingly wide variety of mathematical problems. For example, the *complex singularity index* or *log-canonical threshold* come up, among other places, in:

(1) structural and classification problems arising in the minimal model program;
(2) questions of birational rigidity;
(3) constructions of Kähler Einstein metrics on Fano manifolds;
(4) the theory of multiplier ideals;
(5) Hodge-theoretic invariants of singularities;
(6) constructions arising from the theory of $\mathcal{D}$-modules
(7) Igusa's theory of p-adic zeta functions;
(8) the geometry of the spaces of arcs on a singular variety; and
(9) invariants arising from tight-closure methods in positive characteristic commutative algebra.

While some of the connections among these areas were well understood, the appearance of the same invariants in so many different areas is surprising and suggests that there are some deeper links behind the picture. There are also many outstanding important open problems about these invariants remain open. The solution of these problems would have important applications to the minimal model program and the geometry of the moduli spaces. One of the main goals of our workshop was to bring mathematicians who are working on the different aspects of these invariants together to learn and communicate with each others.

Turning to a more detailed description of the workshop, we tried to have two expository talks each morning. In the afternoon we had sessions on working out basic examples and discussing open problems. Several times we also broke into smaller groups for further discussions.

Mustaţă gave an introductory talk on log-canonical thresholds and minimal log-discrepancies of pairs of the form (X, Y), where X is a variety with mild singularities and Y is a closed subscheme of X. Veys gave an elementary talk on p-adic and motivic integration. He explained how singularities appear in the work of Igusa: if f is a non-constant polynomial in n variables

with integer coefficients, then the number of solutions modulo p^m grows asymptotically like $p^{m(n-c)}$, where c is the log canonical threshold of f.

Phong gave a talk describing some of the work related to the existence of Kähler- Einstein metrics Fano manifolds. He also discussed Yau's conjecture on relating the existence of such a metric to geometric invariant theory. Yau and Tian introduced an invariant of complex manifolds, called the $\alpha-$invariant, using plurisubharmonic functions. If the $\alpha-$invariant of X is greater than $\frac{n}{n+1}$, then Nadel and Tian show that X admits a Kähler-Einstein metric. He also described recent work of Song on approximating $\alpha-$invariants using holomorphic functions. Later in the week, Ross further explained the connections between K-stability and the existence of special metrics.

Hara gave a talk on characteristic p methods. Building on the work of Hochster and Huneke on tight closure, Hara, Smith, Takagi, Watanabe and Yoshida use the Frobenius morphisms to construct analogues of multiplier ideals in positive characteristics. In particular, they introduced a numerical invariant called the F-pure threshold that is an analogue of the log-canonical threshold.

Kawakita gave a talk on inversion of adjunction. He is able to extend the some of the results of Ein, Mustaţă and Yasuda to the case when the ambient space is not a local complete intersections. Using his work he was able to give a partial answer to a question Schwede raised in one of the problem sessions, motivated by trying to connect log-canonical and Du Bois singularities. Alexeev gave a beautiful talk on how to use the higher dimensional minimal model program to study the varieties that appear at the boundaries of the moduli spaces.

One of the most important problems in higher dimensional algebraic geometry is the existence of minimal models of varieties or of pairs. Mori and his collaborators had worked this out for threefolds by proving the existence and termination of flips. Shokurov made many important contributions to this program. In particular, he reduced the existence of general flips to pl-flips which opened the possibility of doing induction on dimension. Using his approach he was able to greatly simplify Mori's proof for threefolds and to prove existence of flips in dimension four.

There was some spectacular progress in the last few years. First, Siu proved the invariance of the plurigenra using multiplier ideals and extension theorems. Combining Shokurov's approach with extension techniques of Siu and Kawamata, last Summer Hacon and McKernan proved the following very important inductive step. They showed that the Minimal Model Program in dimension n implies the existence of flips in dimension $n+1$.

During the workshop, Siu has given a lecture outlining his program on how to use analytic methods to prove directly that the canonical ring of a variety of general type is finitely generated. He expressed optimism that he will be able to carry out the proof in the near future. On the other hand, McKernan has given a talk on joint work in progress with Birkar, Cascini and Hacon. They seem to be very close to proving existence of minimal models for varieties of general type in arbitrary dimension.

Finally we would like to discuss some of the open problems and future directions for work that came up during the workshop. For birational geometry, some of the most interesting conjectures about the numerical invariants are the ACC conjectures of Shokurov. For example,one of these conjectures asserts that if T_n is the set of log canonical thresholds of pairs (X, D), where X is an n-dimensional variety with mild singularities and D is an integral divisor on X, then T_n satisfies the Ascending Chain Condition. Shokurov made

a conjecture about similar invariants, the minimal log discrepancies, and he showed that this would imply Termination of Flips. Together with the inductive result of Hacon and McKernan, this would give the Minimal Model Program in general. It is also possible that one can replace the minimal log discrepancy by a less refined invariant, such that a suitable ACC condition would still give the Minimal Model Program.

For analytic geometry, Siu proposed a program to prove invariance of plurigenera for compact Kähler manifolds. It would also be very interesting to understand the Siu extension theorem and see whether one can apply these techniques to the abundance conjecture and other geometric problems. In positive characteristic, the F-pure threshold is still a very mysterious invariant. The main conjecture in this area is in the context of reducing modulo p a pair (X, Y) from characteristic zero. It is conjectured that for infinitely many primes the F-pure threshold is equal to the log canonical threshold of (X, Y), and it would be very interesting to find conditions that would guarantee the equality.

In the context of p-adic and motivic integration, one of the most important open problems relates the poles of the zeta function with other invariants of the singularities, such as eigenvalues of the monodromy or the Bernstein-Sato polynomial. The picture is well-understood in dimension three, and Veys and others have partial results in higher dimensions. However, the general problem is wide open.

We felt that the workshop was a success. We certainly have learned a lot from the participants about the latest developments in the various areas. With so many outstanding questions and some of the recent spectacular breakthroughs, we are looking forward with excitement to the progress in the next few years. Finally we would like to thank AIM for the financial support and the hospitality.