Small ball inequalities in analysis, probability, and irregularities of distribution

organized by
William Chen, Michael Lacey, Mikhail Lifshits, and Jill Pipher

Workshop Summary

This workshop had researchers from four distinct areas: (1) Irregularity of distributions (2) Probability Theory, (3) Approximation Theory and (4) Harmonic Analysis. In fact, it was the first event that sought to bring together researchers from all of these areas. As such the workshop participants were diverse with respect to area, with many meeting each other for the first time. The principle objective was to overcome barriers of language between these researchers, and share points of view, and different perspectives on a set of closely allied problems. This sharing of view point, vocabulary, and techniques of proof was the main task of the week. Those participants who were new to the subject, be they graduate students, or senior researchers, appreciated the time to acquaint themselves with the leading experts who attended and spoke at this meeting.

The Problem List generated by the workshop grew to an extensive document of more than 30 pages in the end, which touches on all the major subjects of the week.

Irregularities of distribution ([1]) provide an elementary introduction into the problems at hand. Given integer N, and selection $\mathcal{P}$ of N points in the unit cube $[0,1]^d$, we define a *Discrepancy Function* associated to $\mathcal{P}$ as follows. At any point $x \in [0,1]^d$, set

$$D_N(x) = \sharp(\mathcal{P} \cap [0,x)) - N|[0,x)|\,.$$

Here, by $[0,x)$ we mean the d-dimensional rectangle with left-hand corner at the origin, and right-hand corner at $x \in [0,1]^d$. Thus, if we write $x = (x_1, \dots, x_d)$ we then have $[0,x) = \prod_{j=1}^d [0,x_j)$. Thus, at point x we are taking the difference between the actual number of points in the rectangle and the expected number of points in the rectangle. Traditionally, the dependence of D_N on the selection of points $\mathcal{P}$ is only indicated through the number of points in the collection $\mathcal{P}$.

Seminal results here are the universal lower L^2-bound on the Discrepancy function given by Klaus Roth in 1954, that in dimension $d \geq 2$, $\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$. On the other hand in dimension $d=2$, Wolfgang Schmidt in 1972 proved that in dimension $d=2$ that the L^∞ norm of the discrepancy function is always larger than the Roth bound: $\|D_N\|_\infty \gtrsim \log N$. These bounds are known to be sharp.

Bounds for the Discrepancy Function have a deep reflection in the so-called Small Ball inequalities for the Brownian Sheet. The Sheet is a stochastic process that is formed as an appropriate tensor product of the the more well-known one-dimensional Brownian motion. The d-fold tensor product of Brownian motions is the Brownian sheet with $B^d(t_1, \dots, t_d)$ with time parameter in $\mathbb{R}^d$. It can be viewed as an integral of a White Noise measure against rectangles. The Small Ball probability concerns the probability that the Sheet is 'small' with respect to an appropriate norm. For instance, for any d it is known that if we measure the size of the Sheet in L^2-norm we have

$$-\log \mathbb{P}(\|B^d\|_{L^2([0,1)^d))} < \epsilon) \simeq \epsilon^{-2} |\log \epsilon|^{2d-2}\,, \qquad \epsilon \downarrow 0\,.$$

This result is in analogy to Roth's Theorem. In dimension $d = 2$, the result of Talagrand the same question, but with the Sheet measured in L^∞-norm.

$$-\log \mathbb{P}(\|B^2\|_{L^2([0,1)^d))} < \epsilon) \simeq \epsilon^{-2} |\log \epsilon|^{3}\,, \qquad \epsilon \downarrow 0,\ d = 2\,.$$

Underlying this proof is a connection between the stochastic Small Ball Inequalities and a range of approximation issues associated with the reproducing kernel Hilbert space of the associated process, namely Kolmogorov entropy for the unit ball, with respect to an appropriate norm. This space for the Brownian sheet is the Sobolev space of functions of square integrable mixed derivative, but these connections persist for a wide variety of Gaussian processes, see ([3]).

Moreover, to verify these approximation theory properties for the Sheet, Talagrand proved a surprising reverse-triangle inequality for Haar functions adapted to rectangles of a given size, see (‡), which inequality is closely related to Schmidt's Theorem.

The conference included introductory lectures touching on all four areas, and the well-established connection between Small Ball Probabilities and Approximation Theory. These lectures, given by William Chen, Werner Linde, Vladimir Temlyakov, Wenbo Li, and Dmitriy Bilyk, served to give all the participants a solid introduction in the subject. Many of the afternoon topics drew upon these questions, with detailed discussion of entropy and approximation numbers taking up some discussions, especially as they relate to interesting borderline questions.

Recently, Bilyk, Lacey and Vagharshakyan ([2]) have established new results about Discrepancy Function and Small Ball Problems in all dimensions $d \geq 3$, building upon a result of Jozef Beck. These results are the first improvement in the Roth bound in dimension 4 and higher. Their presentations lead to many questions taken up in afternoon sessions. One of the most pressing is identifying a convincing heuristic for the Small Ball Conjecture, which could be turned into a rigorous proof.

Central to the subject of this workshop are multivariate periodic function f supported on on a so-called 'hyperbolic cross.' For a multivariate function f on $\mathbb{T}^d$, set its Fourier expansion into dyadic blocks by

$$\Delta_{(s_1,\dots,s_d)} f = \sum_{\substack{n \in \mathbb{Z}^d \\ 2^{s_j} \leq n_j < 2^{s_j+1}\,,\, 1 \leq j \leq d}} \widehat{f}(n)\, \mathrm{e}^{in\cdot x}\ .$$

The function f has support in the nth hyperbolic cross iff $\Delta_s f \neq 0$ only for $\|s\|_1 = n$. A heuristic which could provide a framework to find a solution to these questions is the QC-norm, invented by Boris Kashin and Temlyakov. It is formed by randomizing the Fourier expansion of such a 'hyperbolic cross' function in the following manner: $\|f\|_{QC} = \mathbb{E}\Big\| \sum_{s\,:\,\|s\|_1 = n} \pm \Delta_s f \Big\|_\infty$. The Small Ball inequality holds for the QC norm, namely

$$n^{(d-2)/2} \|f\|_{QC} \gtrsim \sum_{s\,:\,\|s\|_1 = n} \|\Delta_s f\|_1\,.$$

A combinatorial form of the Small Ball Inequality has an analogous flavor: Let h_R be an L^∞-normalized Haar function adapted to dyadic rectangle $R \subset [0,1]^d$. The conjecture is:

$$n^{(d-2)/2} \Big\| \sum_{|R|=2^{-n}} a_R \, h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |a_R| \,. \qquad (\ddagger)$$

In all dimensions, if the exponent $(d-2)/2$ on the left is replaced by $(d-1)/2$, the inequality is straight forward, by appealing to an average case analysis. This corresponds to Roth's estimate. Notice that in dimension $d \geq 2$, a point x will be in approximately n^{d-1} rectangles. When $d = 2$, the exponent on the left is zero, and one should find a point x where there is no cancellation in the sum. But in case of $d = 3$, each point x is in approximately n^2 rectangles, and the truth of the estimate depends upon selecting an x where, not that all the corresponding Haar functions have the same sign, but that the gap from being equally distributed among plus signs and minus signs is about $\sqrt{n}$. There has been a recent improvement of the average case exponent $(d-1)/2$ to $(d-1)/2-\eta$, for some positive η, which is a function of dimension.

While these new estimates are pleasing with dimension fixed, if one considers the best known estimates as dimension tends to infinity, they are completely unsatisfactory. There are abundant problems concerning optimal, or even improved estimates for either the Discrepancy function, or Small Ball Probabilities, in the situation where dimension is large. Mikhail Lifshits' talk at the Workshop focused on a particular aspect of this for the Brownian sheet. And a problem, with a monetary prize, of a similar nature for the Discrepancy function was posed in the problem session of the Workshop.

The partial resolution of this conjecture in the case of dimension 3 and higher identifies new properties of 'hyperbolic sums' of Haar functions. Namely, the estimate above is Talagrand's inequality in the case of dimension $d = 2$. The continuous analogs of these have yet to be established. This subject was discussed, and the Problem list generated during the conference reflect these discussions.

The connections between Small Ball probabilities and approximation theory are rather precise, whereas the connections Small Ball Probabilities between the Brownian Sheet and the central results of Irregularities of Distribution seem to be merely very close. Nevertheless, there is a crude dictionary between the sets of results here. The possibilities of extending this dictionary to other settings attracted attention during the Workshop. Some attractive problems were identified. One of these concerns a circle of results of Beck, and Beck-Chen, on irregularities of distribution with respect to other geometric figures, such as all convex sets in the unit cube. The analogous Gaussian process would be generated by a White Noise measure integrated against convex sets. The Large Deviations for this process are well understood, while the Small Ball Probabilities are not. Can the insights from Irregularities of Distribution be used shed light on the probabilistic questions? Answers here would extend the reach of the existing dictionary.

In the subject of Irregularities of Distribution, a range of constructions of collections of points which are extremal with respect to the Roth bound are known. While these constructions are relatively simple in two dimensions, the subject in three and higher dimensions becomes quite intricate. These constructions should provide some insights into the probabilistic side, a subject which was taken up in afternoon discussion.

A new question concerning the irregularities of distribution according to rectangles in the unit square, rotated by a set of angles. This question has points of interest from several viewpoints, as analogous questions about Maximal Functions in the plane are very well studied. In the setting of Discrepancy functions, one can anticipate issues related to Diophantine properties of the rotations to arise. Afternoon discussions lead to preliminary results on this question.

On the probabilistic side, there are range of interesting Gaussian processes, such as the fractional Brownian sheets, which are fractional integrals of White Noise measure, or Liouville processes. The Small Ball Probabilities in dimension three merit consideration. Certain of the Liouville processes are boundary cases from entropy considerations. For instance, the operator

$$\mathrm{V}\, f(t) := \int_0^t \frac{f(s)}{\sqrt{t-s}|\log t-s|}\, ds$$

maps $L^2[0,1)$ compactly into $C[0,1]$. Appropriate upper bounds on the Kolmogorov entropy do not follow from abstract considerations. An argument which sheds light on this bound would shed additional light on such boundary cases.

Small groups discussed these problems. In a different direction, one can consider instead of Gaussian processes, stable processes, or even Levy processes. These processes arise in canonical ways, like the Gaussian processes, but present significant new difficulties in their analysis, in ways that make their analysis more subtle and refined. Frank Aurzada spoke about these processes in the morning, with the issues raised in this talk forming discussions for remainder of the week.

There are additional two grand questions in the subject of Irregularities of Distribution, which do not seem to have a useful analog on the Probability side. The point of interest is three and higher dimensions, as the two-dimensional case is understood. The first is nature of the optimality of the Small Ball Inequality lower bound on the L^∞-norm on the Discrepancy function. Recently it was proved that

$$\|D_N\|_\infty \gtrsim (\log N)^{(d-1)/2+\eta(d)}, \qquad d \geq 3$$

for some positive $\eta(d) > 0$. On the other hand, the most that one could hope to prove using properties of hyperbolic Haar functions is the estimate $\|D_N\|_\infty \gtrsim (\log N)^{d/2}$. This is in contrast to the best known constructions of sequences which are small with respect to these metrics. One can construct sequences with $\|D_n\|_\infty \lesssim (\log N)^{d-1}$. The true nature of this question was the subject of lively discussion. Those attending could not agree on the any single conjecture, with alternates being posed. Potential paths to lower the best known estimates were also the subject of discussion.

The second grand question concerns the L^1-norm of the Discrepancy function. The principal conjecture here is that the L^1-norm of the Discrepancy Function satisfies the L^2-bound: $\|D_N\|_1 \gtrsim (\log N)^{(d-1)/2}$. This is true in dimension $d = 2$, thanks to a remarkable Riesz product argument of Halász. There is no extension of this result to higher dimension, a question that has been the subject of a lot of unsuccessful speculation over the years. This subject also occupied a good deal of speculation. There are reasonably concise explanations of the difficulty of extension, but the way around them is still not clear.

Bibliography

[1] Jzsef Beck and William W. L. Chen, *Irregularities of distribution*, Cambridge Tracts in Mathematics, vol. 89, Cambridge University Press, Cambridge, 1987.MR903025 (88m:11061)

[2] Dmitriy Bilyk, Michael T. Lacey, and Armen Vagharshakyan, *On the small ball inequality in all dimensions*, J. Funct. Anal. 254 (2008), no. 9, 24702502. MR 2409170

[3] Wenbo V. Li and Werner Linde, *Approximation, metric entropy and small ball estimates for Gaussian measures*, Ann. Probab. 27 (1999), no. 3, 15561578.MR 2001c:60059