Small scale dynamics in incompressible fluid flows

organized by

Tarek Elgindi, Aseel Farhat, Anna Mazzucato, and Wojciech Ozanski

Workshop Summary

Workshop Focus

Fluid dynamics refers to an interdisciplinary effort aimed at understanding the complex behavior of fluids, particularly the phenomenon of turbulence. Developing a rigorous theory consistent with experiments, computational simulations, and the observed phenomenology of turbulence has remained a significant challenge in the mathematical fluids community since the foundational works of Kolmogorov, Onsager, and Taylor in the 1930s-1950s. Even a precise definition of turbulence remains elusive.

However, it is anticipated that the emergence of turbulent flows is associated with the dominant role of small-scale dynamics in fluid flows. This association may be related to singularities near a boundary of a domain, putative flow instabilities, turbulent layers, self-similar blow-up profiles, and other singularities. Remarkably, some turbulent phenomena are observable through aspects of the well-posedness theory of partial differential equations (PDEs) arising in fluid mechanics.

Recent developments have shed light on the chaotic behavior observed in turbulent flows, the role of criticality in ill-posedness results, singularity formation in finite time of classical solutions, and the relationship between irregularity, instability, and non-uniqueness of solutions. Many exciting results have been published in the field of mathematical fluid dynamics in recent years, with contributions from invited participants and organizers of this workshop.

The primary goal of the workshop was to bring together leading experts in the mathematical theory of hydrodynamics models and stimulate extensive discussions on recent developments and possible future research directions. The main focus of mathematical activities in the workshop was the investigation of small-scale dynamics in fluid flows and their role in emerging singularities and turbulent flows. The objective was to develop a rigorous theory consistent with computational simulations and the observed phenomenology of turbulence. Main topics of interest included ill-posedness and singularity formation in fluids, hydrodynamical instabilities and turbulent mixing, mathematical analysis of fluid boundary layers, and inviscid limits and anomalous dissipation in fluids. The workshop featured talks introducing the main ideas and problems in each topic, and working groups were formed during the week to address the problems outlined below. We anticipate that the workshop will lead to several new and interesting research directions, fostering continued communication.

Problem Studied in Working Groups

Enhanced and Anomalous Dissipation of the 2D NSE.

The initial problem (proposed by Tarek Elgindi) discussed in this group was to study the phenomena of enhanced dissipation and anomalous dissipation of the 2D incompressible Navier–Stokes equations in a more general setting. The overall idea is to first construct a low-regularity steady-state solution of the 2D Euler equations, take it as the initial data for the corresponding Navier–Stokes equations, and then analyze how the energy/enstrophy dissipation behaves in the zero-viscosity limit. Ideally, if the underlying Euler steady-state solution is sufficiently irregular, the energy/enstrophy dissipation shall tend to zero much slower that in the smooth case as viscosity vanishes. As agreed by the whole group, stationary vortex patch solutions to the Euler equations would be nice candidates for such purpose. This type of steady-state solutions are easy to construct in a 2D tunnel. Based on a particular periodic vortex patch solution, the group performed some preliminary analysis and showed that enhanced dissipation and anomalous dissipation are possible in such scenario. However, to come up with a conclusive result, it still requires some further analysis and calculations. The group agreed to leave this problem to future discussions.

In the last two days of the workshop, the focus of the project turned to another related problem (an open question): Is it possible to construct a stationary vortex patch solution to the 2D Euler equations in the half-space with a constant background flow? A promising idea popped up quickly after some insightful discussions. That is, by properly parameterizing the boundary of the vortex patch, one can reformulate the problem equivalently into a fixed-point problem involving a nonlinear integral map. It then reduces to proving the existence of a fixed point of this map. To test this idea, the group performed some numerical computation on the fixed-point problem and observed that fixed-point iteration converged stably to a potential solution, which inspired confidence in solving the problem analytically. As an extension of the workshop, the group is now actively working together on this fixed-point problem.

Group members: Alexey Cheskidov, Elaine Cozzi, De Huang, In-Jee Jeong, Piotr Kokocki, Farjana Siddiqua, and Peicong Song.

Non-existence of self-similar solutions of the 3D NSE in L^3.

For PDEs which possess a natural scaling, it is natural to look for blow-up solutions which are self-similar, i.e., scaling invariant. For the 3D Navier-Stokes equations, this was mentioned by Jean Leray in his pioneering work in the 1930s. In the 1990s, the existence of non-trivial self-similar blow-up solutions on the whole space was ruled out by Necas, Ruzicka and Sverak when the similarity profile is in the critical Lebesgue space. This was later improved by Tsai to exclude blow-up scenarios under weaker decay assumptions. The proofs delicately rely on a stationary system satisfied by the self-similar profile. Part of this system, the total head pressure, enjoys a maximum principle which, alongside decay properties for the standard pressure, implies the energy of the profile is zero. On the half-space, the result of Necas, Ruzicka and Sverak is known by other methods, but that of Tsai is not. The problem is that, although the total head pressure appears to still enjoy a maximum principle, the pressure does not vanish on the boundary, indicating the total head pressure may be positive. This possibility breaks Tsai's argument.

One working group began to examine this problem. The group found several reasons for optimism. First, the group analyzed a toy 1D version of the elliptic inequality and found that non-trivial solutions must be singular at the origin, violating the finite boundary condition requirement for the total head pressure. Second, in the original proof of Necas, Ruzicka and Sverak, there is room for a sign varying term, so long as it can be compared to a fraction of the energy of the similarity profile. Third, the main issue lies in the boundary values for the pressure which may be non-zero. Until recently, the boundary conditions for the pressure were not well understood. Recent work of Maekawa, Miura and Prange has resolved this, providing new tools compared to those available to researchers previously working on this problem. These observations provide a foundation for further research on this topic by the group members.

Group members: Zachary Bradshaw, Hussain Ibdah, Stan Palasek, Hao Jia, and Wojciech Ozański.

Norm inflation for 2D Euler, for $u \in H^s, s \in (0,1)$.

During discussions it was noted that the main difficulty is the very low required regularity of the velocity field, which contrasts with the known conservation laws for both the L^2 and H^1 norm of the velocity. A priori, it is not clear what mechanism could be investigated for growth of the solution. For example, one can consider a similar scenario as in the more regular cases (such as in the recent result on the instantaneous gap loss of Sobolev regularity for $s \in (1,2)$). Although this could be a promising candidate, some important improvements to the method would be necessary to obtain sufficient control of the local-in-time behaviour of the solution. One possible approach could combine techniques from prior work on SQG, where the H^s bounds are obtained directly, rather than by making use of the particle trajectories. However, in analogy to the approximation used in the SQG case, this would require, roughly speaking, an infinite order of approximation.

Group members: Diego Cordoba, Slim Ibrahim, Luis Martinez-Zoroa, Nader Masmoudi, and Wojciech Ozański.

Rate of Growth of Line Segments under 2D Euler flow.

The group worked on constructing a 2D Euler flow such that there exists a line segment of unit length at time zero, and the length would grow exponentially in time for all time carried by the flow, while still remaining in a bounded domain. Essentially we want the line segment to twist and fold upon itself repetitively.

During the workshop, we came up with the idea of using point vortices (and its associated smooth approximations) for the Euler flow. We use computer programs to sanity check whether numerically it would be exponentially growing. One vortex would have exactly linear growth, and two vortices also have approximately linear growth, to our expectations. The case of three interacting vortices with equal strength seemed to generate exponential growth of the line segment if the initial line crosses the trajectory of the vortices, and such exponential growth appears very robustly with respect to initial configuration of the location of vortices and line segment. We are working on understanding it and justifying it rigorously.

Group members: Slim Ibrahim, Yixuan Wang, Helena Nussenzveig Lopes, and Tarek Elgindi.

Boundary layers and the vanishing viscosity limit with injection and suction.

The group studied the creation of boundary layers and the validity of the vanishing viscosity limit with injection and suction boundary conditions. In general, the validity of the vanishing viscosity limit, that is, whether solutions of the Navier-Stokes equations converge to solutions of the Euler equations if the viscosity vanishes, is unknown. It is know, however, that injection and suction can stabilize the layer, but a rigorous analysis has only been perormed in specialized settings, namely, channels with injection and suction normal to the wall or when the flow is linearized around a constant shear. The group started out studying the case of flows in an annulus, to see whether the concavity and convexity of the boundary plays a role. It then reverted to the case of a channel, in fact, localized to a half-plane with non-trivial suction. They investigated whether the vanishing viscosity limit can be proven using the recent techniques of Vasseur and Yang to estimate the rate of separation in the boundary layer, when the normal suction is stronger than the tangential shear at the boundary. They realized that these techniques are not sufficient to establish the vanishing viscosity limit, and may require the use of a corrector, but allow to estimate the rate of energy dissipation as viscosity vanishes and the rate of separation in the boundary layer as a function of time. As a byproduct they obtain a local bound on the vorticity production at the boundary. This work is ongoing.

Group members: Anna Mazzucato, Alexis Vasseur, Jincheng Yang, Vincent Martinez, and Aseel Farhat.