Sparse domination of singular integral operators

organized by
Amalia Culiuc, Francesco Di Plinio, and Yumeng Ou

Workshop Summary

The workshop on Sparse domination for singular integral operators met at AIM between October 9 and October 14 2017. During the morning sessions, ten participants gave talks on applications of sparse domination principles in various areas, namely

(1) Rough singular integrals (Andrei Lerner)
(2) Matrix weighted estimates (Stefanie Petermichl and Sergei Treil)
(3) Spherical averages and Radon transforms (Michael Lacey and Laura Cladek)
(4) Sparse domination in homogeneous and non-homogeneous settings (José Manuel Conde Alonso and Grigori Karagulyan)
(5) Multiparameter sparse bounds (Alexander Barron)
(6) Weighted estimates for discrete operators (Pavel Zorin-Kranich and Luz Roncal)

These directions were present in the Monday afternoon problem session, moderated by Tuomas Hytönen, when 31 questions were proposed for investigation. Throughout the week, the participants focused on six particular topics and interrelated problems.

(1) **Group 1: Sparse bounds via sets better adapted to the geometry of the spherical maximal average and Bochner-Riesz multipliers**
Participants: Alexander Barron, Michael Lacey, Teresa Luque, Yumeng Ou, Betsy Stovall.
Group 1 focused on exploring possible sparse bounds for the spherical maximal operator and Bochner-Riesz multipliers that are based on sets better adapted to the geometry of the operators. Recently, the traditional sparse bounds (based on cubes) have been studied for these operators by Lacey et al., but it is expected that sparse forms with respect to other basic elements could provide better quantification, as geometry plays a key role in the study of these operators.
The group met for one day and brainstormed possible ways to formulate the question. One of the candidates in the study of the lacunary spherical maximal operator are dual ellipsoids on the sphere, which are known to dictate the behavior of the single scale averaging operator in a certain sense. And curved tubes or parallalelopipes come into play in a similar way if one studies a closely related object: singular integrals along moment curves. The group also discussed progress made towards sparse bounds of Bochner-Riesz multipliers using tilted rectangles, and possible challenges and counterexamples that may occur with this approach. The group is still in communication and expects to continue to work on sparse bounds using dual ellipsoids.
(2) **Group 2: Sparse bounds for the strong maximal function and singular integrals in the Zygmund dilation setting**

Participants: Alexander Barron, Laura Cladek, Grigori Karagulyan, Teresa Luque, Virginia Naibo, Anh Neuman, Yumeng Ou, Rodolfo Torres.

The group had rotating members and have worked on several problems related to sparse bounds of multiparameter singular integrals.

One of the main focuses of the group was sparse bounds of the strong maximal function M_S and its sharp weighted estimate. The group spent a lot of time trying to construct a counterexample that disproves the existence of such a sparse bounds (defined in the classical sense but with respect to axes-parallel rectangles). Many ideas were discussed and some progress has been made, and the group will continue this line of investigation. On the other hand, the group also considered consequences in quantitative weighted estimates for M_S assuming that the sparse bounds exist, and reduced the problem to the study of the weighted strong maximal function. In fact, even though the group have come up with possible ideas for counterexamples, they also found positive evidences indicating that the sparse bounds could still hold. More precisely, the group successfully showed that the sparse bounds would be compatible with the mapping properties of M_S in the sense that it would recover the bound $M_S : L\log L \to L^{1,\infty}$ and would not imply the impossible bound $M_S : L^1 \to L^{1,\infty}$.

Moreover, the group also studied sparse bounds for singular integrals that are invariant under Zygmund dilations $(x, y, z) \mapsto (sx, ty, stz)$, and found that one can prove a sparse bound for Haar multipliers in this setting via sparse forms involving Zygmund square functions. The proof proceeds very similarly as in the biparameter setting that was studied by Barron and Pipher.

(3) **Group 3: Weighted estimates for rough singular integrals**

Participants: David Cruz-Uribe, Andrei Lerner, Kabe Moen, Carlos Pérez.

The group considered the problem of establishing a sparse domination approach towards sharp weak type estimates for the rough homogeneous singular integral T_Ω. It was conjectured that T_Ω satisfied an $(L\log L, L^1)$ sparse bound. The proof of this result depends on the weak (1,1) boundedness of a certain maximal function, which is still under investigation. Some progress has been made in that this maximal function estimate has been further reduced to the analogue for a seemingly easier sparse operator.

(4) **Group 4: Sparse estimates for discrete operators**

Participants: Shaoming Guo, Michael Lacey, Ioannis Parissis, Pavel Zorin-Kranich.

Group 3 focused on the study of the discrete maximal operator $\sup_N \left| \dfrac{1}{N} \sum_{n=1}^{N} f(m+n^2) \right|$ and the problem of establishing sparse control. It was determined that a sparse version of Bourgain's log lemma would be useful in this setting, and the participants are currently investigating this potential result.

(5) **Group 5: Sparse domination for the variation norm lacunary spherical maximal operator**

Participants: David Beltran, Shaoming Guo, Richard Oberlin, Betsy Stovall.

The group studied variation norm lacunary spherical maximal operator and proved its sparse bound in certain range using interpolation and ideas from Lacey's work on the lacunary spherical maximal function. In order to find the sharp region where sparse bounds hold true, the group then reduced the problem to the study of the optimal $L^p \to L^q$ improving estimates for the single scale operator in the variation norm setting, which is a very interesting open question.

(6) **Group 6: Matrix weighted theory**

Participants: Amalia Culiuc, Tuomas Hytönen, Dario Mena-Arias, Maria Cristina Pereyra, Stefanie Petermichl, Sergei Treil, Alexander Volberg.

An overarching question in matrix weighted theory is whether the norm of a Calderón-Zygmund operator acting on a matrix weighted L^2 space depends linearly on the A_2 characteristic. In particular, one may ask if such an estimate would hold for specific operators, such as Haar shifts or the Hilbert transform. In light of the recent result of Hytönen, Petermichl and Volberg, which proved a linear upper bound for the dyadic square function, Group 6 focused on the problem of producing a sparse domination proof of the lower bound, first in the scalar case, and then on vector-valued function spaces.

The group was successful in finding a scalar sparse proof and worked on adapting the result to matrix weights. In the process, other questions arose, such as potentially extending the argument to the biparameter setting. The group is still in communication and some of the participants have planned to meet within the upcoming month to continue work on the project.

(7) **Group 7: Sparse domination in non-homogeneous spaces**

Participants: José Conde Alonso, David Cruz-Uribe, Francesco Di Plinio, Stefanie Petermichl, Sergei Treil, Alexander Volberg, Pavel Zorin-Kranich.

The broad concern of this group was to establish a satisfactory sparse domination principle for Calderón-Zygmund operators acting on non-doubling measures $\mathrm{d}\mu$. A characterization of weights w such that standard CZ operators T are bounded on $L^2(w\mathrm{d}\mu)$ is due to Tolsa: the necessary and sufficient condition is that a certain maximal operator M maps to $L^2(w\mathrm{d}\mu)$ to itself boundedly This group discovered that this maximal operator involving modified averages can be discretized and dominated by a sparse operator involving usual averages. This offers a new sufficient condition on w of A_2 type for the boundedness of M (and therefore of general CZ kernels on $L^2(w\mathrm{d}\mu)$. This is an interesting result which can (and will) serve as a starting point for ongoing investigations of the participant. Further attempts of modifying the known method to obtain a sharper (compared to the ones known up to the workshop due to Conde and Parcet, and to Volberg and Zorin-Kranich) sparse domination result directly for T did not lead to significant progress. The problem remains under investigation.