Spectral graph and hypergraph theory: connections and applications

organized by
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Workshop Summary

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Introduction

This is the workshop summary of the AIM workshop *Spectral Graph and Hypergraph Theory: Connections & Applications*, December 6-10, 2021.

Talks

We had one or two talks in each morning session and working sessions in the afternoon. The speakers and their titles are given in Table 1.

Monday	Yufei Zhao	*Second eigenvalue multiplicity*
	Yuval Filmus	*Intersecting families and Hoffman's bound*
Tuesday	Gregory Berkolaiko	*Universality of nodal count statistics in large graphs*
	Mike Tait	*Extremal questions in spectral graph theory*
Wednesday	Cynthia Vinzant	*Log-concavity and the geometry of polynomials*
	Krystal Guo	*Counting walks and eigenvalues*
Thursday	Joshua Cooper	*Homogeneous eigenvalues of a hypergraph and their associated eigenvarieties*
	Elizabeth Yang	*Local spectral expansion and random walks on high-dimensional expanders*
Friday	Pepijn Wissing	*Complex unit gain graphs and what they might represent*

Table 1. Talks given during the workshop.

Working groups

Based on the list of open problems from the problem session and which arose during the morning talks, various groups were formed to work on some problems. In this section, we describe the working groups. The groups described have plans to meet after the workshop and continue work on these projects.

Multiplicity of λ_2 for Cayley graphs of bounded degree.
Members: Xiaofeng Gu, Theo McKenzie, Irene Sciriha, Sam Spiro, Nikhil Srivastava, Michael Tait, Yufei Zhao.

Let G be a graph with maximum degree bounded by Δ.

(1) For the following choices of Hermitian matrix M_G associated with G, what is the maximum multiplicity of the second-largest eigenvalue of M_G?

(a) The adjacency matrix: in this case a sublinear upper bound was proved by [JTYZZ21], with an improvement in the case of regular graphs provided by [MRS21] and also for the normalized adjacency matrix of general bounded degree graphs. Constructions of graphs establishing lower bounds were given by [HSZZ21].
(b) The Laplacian matrix.
(c) Schrödinger operators: any matrix of the form $D+A_G$ where A_G is the adjacency matrix and D is an arbitrary diagonal matrix.
(d) Weighted adjacency matrices: any Hermitian matrix M_G such that $M_G[i,j]=0$ if $\{i,j\} \notin E(G)$.

(2) For a fixed choice of parameters ρ and Δ, what is the largest possible multiplicity of the second eigenvalue of the adjacency matrix A_G with the constraint that $\lambda_2(A_G) = \rho$.

The group investigated the largest multiplicity of the second eigenvalue of the adjacency matrix of Cayley graphs [lee2008eigenvalue].

Jordan blocks of the non-backtracking matrix.
Members: Gregory Berkolaiko, Jane Breen, Kristin Heysse, Kate Lorenzen, Carolyn Reinhart, Xinyu Wu.

This group studied the size of the largest Jordan blocks in non-backtracking matrices for non-regular graphs. For regular graph, it is known that the Jordan blocks of the Jordan decomposition of the non-backtracking matrix have size at most 2 [LubPer2016], but this does not seem to be the case for non-regular graph. The group studied possible generalizations of the results from [GloKem2021].

Small computations in SageMath suggest that graphs with repeated factors in their minimal polynomials have many lots of three cycles. Inspired by this and other computation data, the group plans to explore the connections between eigenvectors and generalized eigenvectors of the non-backtracking matrix and combinatorial structures in the graph.

Positive and negative squared energies.
Members: Aida Abiad, Leonardo de Lima, Dheer Noal Desai, Krystal Guo, Leslie Hogben, José Madrid and Sam Spiro.

Let G be a graph with eigenvalues $\{\lambda_i\}_{i=1}^n$. We consider the *square positive energy* of G, denoted $s^+(G)$, defined as follows:

$$s^+(G) = \sum_{\lambda_i>0}^{r} \lambda_i^2.$$

Similarly, the *square negative energy* of G, denoted $s^-(G)$, is defined as follows:

$$s^-(G) = \sum_{\lambda_i<0}^{r} \lambda_i^2.$$

This project aims to solve the following conjecture which was raised in [ElpGolFar2014]:

Conjecture 0.1. [ElpGolFar2014] *If G is a connected graph of order n, then $s^+(G) \geq n-1$ and $s^-(G) \geq n-1$.*

During the AIM workshop working sessions, the group obtained preliminary results; we have established the conjecture for graphs with an induced bipartite subgraph with at least $n-1$ edges, complete graphs minus an edge, and we showed that $s^+(G) \geq n-1$ for graphs with average degree at least $\sqrt{n-1}$.

The group plans to meet weekly after the AIM workshop to continue working on the project. Future directions include proving the stronger version in Conjecture 0.2 posed by K. Guo during the workshop open problem session, which also generalizes to s^+ a known result for spectral radius, namely that $\rho(G+uv) \geq \rho(G)$.

Conjecture 0.2. *If G is a graph and $uv \notin E(G)$, then $s^+(G+uv) \geq s^+(G)$.*

Other aspects of the project include settling either the original conjecture or the stronger formulation for s^+ for various classes of graphs; long odd cycles, graphs with exactly two positive eigenvalues (using the characterization of [Obo2016]) and graphs with large spectral gap.

Polytopes, lattice lengths, λ-local expanders.
Members: Sebastian Cioabă, Himanshu Gupta, Sidhanth Mohanty, Anurag Sahay, Cynthia Vinzant, Elizabeth Yang.

A long-standing conjecture of Mihail and Vazirani states that the expansion constant of the graph of any $\{0,1\}$-polytope is at least 1. This conjecture has been proved recently [anari2019log] for the matroid polytope (the convex hull of the indicator vectors of the bases of the matroid) and the proof was obtained using an inequality connecting the spectrum of this graph to the spectrum of certain high-dimensional objects. Our group tried applying these methods involving the Alev-Lau inequality [levi2020improved] to study the spectral gap of the associahedron graph and a conjecture of Aldous related to this gap [mcshine1999mixing,molloy1999mixing].

Spectral Turán problems.
Members: Joshua Cooper, Dheer Noal Desai, Leslie Hogben, Himanshu Gupta, Anurag Sahay.

This group surveyed the literature surrounding spectral Turán problems for graphs and considered the question of maximizing the spectral radius for C_{2k}-free graphs [Nik2010], for the recently settled case of hexagon-free graphs [ZhaLin2020], and minimizing the spectral radius for K_t-saturated graphs [Nik2021].

The group eventually decided to consider the spectral analogue of hypergraph Turán problems after the workshop. In particular, their aim is to investigate the spectral analogue of the K_4^3 hypergraph Turán problem: maximizing the spectral radius of a K_4^3-free 3-uniform hypergraphs on n vertices.

Counting hypergraph substructures with characteristic polynomials.
Members: Joshua Cooper and Krystal Guo.

Clark and Cooper [ClaCoo2021] give a hypergraph analogue of the Harary-Sachs determinantal expansion of the characteristic polynomial of a graph. The original Harary-Sachs expansion has a combinatorial interpretation using heaps of pieces [Vie1989]. The group will continue to meet to apply these ideas to hypergraphs, give combinatorial interpretations of generating functions on hypergraphs arising from characteristic polynomials, and explore ideas about hypergraph analogues of the Reconstruction Problem.

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