Stability in mirror symmetry

organized by
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Workshop Summary

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Organizers: Tristan Collins and Jason D. Lotay

25 – 29 April 2022

Introduction.

This workshop focused primarily on two areas of geometry which both play a key role in the study of mirror symmetry: minimal Lagrangians and Lagrangian mean curvature flow (LMCF) on the one hand, and deformed Hermitian-Yang–Mills (dHYM) connections on the other. Minimal Lagrangians and dHYM connections are related by mirror symmetry of Calabi–Yau manifolds and their existence is conjectured to be governed by a stability condition. As a consequence, stability conditions are expected to play a crucial role in understanding the behaviour of geometric flows which have minimal Lagrangians and dHYM connections as critical points, which is LMCF in the case of minimal Lagrangians. Mirror symmetry beyond the Calabi–Yau setting, both to Kähler–Einstein manifolds with definite first Chern class, and to certain non-Kähler settings, were also important areas for discussion in this workshop.

The workshop brought together a diverse range of researchers, both specialist in the area and in interrelated topics, and there was a good mix of junior researchers (graduate students and postdocs) and established researchers. As a consequence, there was a great interplay between the participants, each bringing their own interests, knowledge and ideas, and there was a lot of lively discussion.

Summary.

The main topics discussed in the workshop were as follows.

- LMCF and minimal Lagrangians: relation to Bridgeland stability conditions; singularity formation and long-time existence; soliton solutions; uniqueness questions.
- DHYM connections: relation to GIT and stability; existence and non-existence results; generalizations of dHYM, including to higher rank and complexes of vector bundles.
- Mirror symmetry: obstructions to minimal Lagrangian sections from dHYM; toric mirror symmetry; mirror symmetry and stability in the non-Kähler setting.

Each day, as per the usual AIM format, the workshop consisted of two talks followed by problem sessions, as well as allowing for time for informal discussion.

Talks.
The talks served both to provide background and surveys, as well as give more specific talks on results, including showcasing of work by junior participants.

Pranav Pandit described the general framework of "categorical Kähler geometry", which is motivated by homological mirror symmetry. Pranav explained how this formalism provided a prediction for the behaviour of LMCF.

Andrew Hanlon described how to derive an osbtruction to the existence of certain special Lagrangians in the SYZ mirror to toric varieties. This obstruction arose from considerations of dHYM connections and mirror symmetry. Andrew also gave some open problems arising from this work.

Felix Schulze discussed singularity formation in LMCF and the role of ancient solutions. Felix described clasification results for ancient solutions of LMCF, including recent work related to translators which arose from the previous online version of this AIM workshop (and answered an open problem from that workshop).

Abigail Ward described a version of homological mirror symmetry for some non-Kähler surfaces, including Hopf surfaces.

Tommaso Pacini discused links between ambient curvature conditions and log-convexity of the volume functional defined on orbits of a group action, particularly related to (minimal) Lagrangian submanifolds and Kähler geometry. Tommaso also discussed links between plurisubharmonic functions in calibrated geometry, calibrated fibrations and subharmonic/convex functions.

Tristan Collins gave a survey of known existence and obstruction results for the dHYM equation, particularly in the hypercritical phase regime. Tristan discussed the analogue of the Kempf–Ness functional, geodesics in certain non-positively curved spaces, as well as stability conditions.

Adam Jacob described his recent work on the dHYM equation with Calabi-symmetry. Adam described in this concrete setting how the stability conditions and obstructions from the general theory arise and are related to the existence/non-existence of dHYM connections.

John McCarthy gave an overview of his PhD thesis work on Z-critical connections (which generalize dHYM connections) and stability conditions. John linked his talk to previous ones and raised the key question about the link between Z-critical connections and mirror symmetry.

Mario Garcia Fernandez described the $\frac{1}{2}$-twisted model which arose in some of the original physics literature on mirror symmetry, which is valid in non-Kähler settings. Mario then described a result showing a version of mirror symmetry for the $\frac{1}{2}$-twisted model for certain Hopf surfaces. Mario also discussed the role of stability in this non-Kähler mirror symmetry.

Chris Woodward gave a discussion of Lagrangian branes and the quantum minimal model program, with links to mirror symmetry and LMCF, including the Joyce program for the

Thomas–Yau–Joyce conjecture. Chris also raised interesting open questions related to Lagrangian surgeries and Lagrangian Floer homology.

Problem sessions.
The problem sessions went very well, with participants exploring the different topics, having engaging discussions, and forging new academic relationships. The main problems discussed were as follows.

(1) **What are the dimensional reductions of 4-dimensional dHYM?** The group derived the (local) equations for the dimensional reduction from 4 to 2 dimensions along a Lagrangian fibration. The group showed that for a compact surface the equation reduced to the Hitchin system, but for punctured surfaces one might expect new solutions (based on ideas proposed by T. Collins and A. Jacob), which the group planned to explore. The group also derived the the analogue of the ADHM construction of instantons on $\mathbb{R}^4$ and looked for solutions, which mainly reduced to the instanton construction. The group also considered the dimensional reductions to 1 and 3 dimensions (i.e. the analogue of Nahm's equations and monopoles) but ultimately decided these reductions were unnatural in this context given the preferred complex structure in the dHYM equations.

(2) **Can one prove the Thomas–Yau–Joyce conjecture for the flat 2-torus?** The group showed that the conjecture is true for the flat 2-torus but that the Joyce program for proving the conjecture through LMCF was not just the usual curve shortening flow proof. The group argued that the proof reduced to a combinatorial problem in identifying which curves in the 2-torus are graded and unobstructed. This was a great example of a very fruitful dialogue between researchers from diverse communities: in this case, geometric analysis and symplectic topology. The group plans to continue to try to prove the conjecture.

(3) **Are there finite-time singularities for the dHYM flow or graphical LMCF?** The group studied the case of the dHYM flow in the Calabi-setting described in detail by A. Jacob. The group showed that the flow in this case reduced to modified curve shortening flow, and showed that there were no finite-time singularities. The group also thought about graphical Lagrangian mean curvature flow more generally, and particularly in the 2-dimensional case, and realized that the key issue seemed to be to ensure that graphicality is preserved. However, key open questions in analysis of both the dHYM flow and graphical LMCF remain, and the group is keen to pursue these questions in the future.

(4) **What can one say about LMCF and minimal Lagrangians in Kähler–Einstein manifolds with definite first Chern class?** This group looked at LMCF for the case of $\mathbb{CP}^n$ and for negative first Chern class. In particular, it seemed promising to consider mean curvature flow (not necessarily for Lagrangians) near minimal Lagrangian in negative Kähler–Einstein manifolds. The group also tried to relate minimal Lagrangians to matrix factorisations and Landau–Ginzburg models via mirror symmetry.

(5) **What is the correct notion of dHYM for higher rank bundles and complexes of vector bundles?** The group used the Euler sequence for $T\mathbb{CP}^2$ (motivated by a Lagrangian multisection on the Landau–Ginzburg mirror to $\mathbb{CP}^2$) to postulate a definition of dHYM for complexes of vector bundles, and which gave a dHYM connection on $T\mathbb{CP}^2$. They studied their definition for certain cases (coming from

Riemann surfaces and quivers), and postulated a Kempf–Ness functional. Finally, the group considered LMCF for the Lagrangian multisection on the mirror of $\mathbb{CP}^2$ and its relation to dHYM.

In summary, important progress was made towards each of these problems and it seems very likely that these discussions will yield concrete progress on both dHYM connections and LMCF.

Conclusion.

The organizers were extremely pleased with the results of the workshop, which greatly exceeded their expectations. As well as the participants learning a lot about the area, there was tangible progress on numerous fronts, new collaborations were developed, and the workshop is likely to lead to significant results for the field in the not-too-distant future. The organizers are grateful to the AIM staff for providing support for this very successful workshop.