Criticality and stochasticity in quasilinear fluid systems

organized by
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Workshop Summary

1. Overview

Mathematical analysis of partial differential equations in fluid mechanics have seen remarkable progress in the last few decades. In its study, "criticality" plays a crucial role in various ways: exponents of the fractional Laplacian in the diffusion above which global well-posedness may be proven; exponents of the initial regularity space above which local well-posedness has been verified, and similarly spatial dimension below which local well-posedness is known; etc. On the other hand, "stochasticity" plays a crucial role in investigation of turbulence, and it has been well documented in recent works that random noise can possess regularizing effects.

A recent ground-breaking result is the non-uniqueness of weak solutions to the three-dimensional Navier-Stokes equations of fluids by Buckmaster and Vicol [BV19] and its technique of convex integration, which was an extension of the previous important works by De Lellis and Székelyhidi Jr. [DS09, DS10, DS13], and remarkably rooted in the work of Nash [N54] in differential geometry. Inspired by such an interdisciplinary breakthrough, this workshop "Criticality and Stochasticity in Quasilinear Fluid Systems" at the AIM was organized to bring together many active researchers in fluid dynamics who possess complementary knowledge and work through different approaches to share their work, suggest open problems, and make significant progress toward future collaborations.

2. List of open problems discussed

2.1 Lectures

The 5-days workshop ran everyday with two talks each morning:

(1) Monday 04/05: In-Jee Jeong and Yao Yao
(2) Tuesday 04/06: Hongjie Dong and Thomas Hou
(3) Wednesday 04/07: Alexey Cheskidov and Matthew Novack
(4) Thursday 04/08: Rongchan Zhu and Hakima Bessaih
(5) Friday 04/09: Hao Shen and Anna Mazzucato

Here are some of the problems which were suggested and discussed:

(1) Various problems about singularity formulation were suggested. First, is there singularity formulation for active scalars (e.g., two-dimensional super-critical surface quasi-geostrophic equations) for which global well-posedness have not been proven starting from smooth initial data? Second, if we reduce the initial regularity from C^∞ to "slightly smooth" (e.g., C^α for finite $\alpha > 0$), can we prove such singularity

formulation? In relevance, another question is whether we can construct solution that grows fast in specific norms; e.g., C^1-norm growth exponentially fast in time.

(2) Various problems concerning patch solutions were suggested. In particular, let Ω denote an angular velocity. Does there exist simply-connected rotating patch for $\Omega \in (0, \frac{1}{2})$?

(3) Global well-posedness for the one-phase Muskat problem, which is a free-boundary problem of two-dimensional fluid filtered in porous media, was recently proven, and its extension to the three-dimensional case was suggested as an open problem.

(4) Various problems concerning ill-posedness via the method of convex integration were suggested. First, can one prove non-uniqueness of Leray-Hopf weak solution to the three-dimensional Navier-Stokes equations diffused via fractional Laplacian with its exponent above $\frac{1}{3}$? Second, can one build Hölder continuous solutions to three-dimensional ideal magnetohydrodynamics system? Third, can one build solutions to two-dimensional surface quasi-geostrophic equations with its solution in $C_t^0 L^p$ for any $p \in [2, \infty]$? Finally, can convex integration be applied to dispersive PDE?

(5) Several open problems about application of convex integration to stochastic Navier-Stokes and Euler equations to prove their ill-posedness were suggested. First, the non-uniqueness results thus far have been achieved only for additive or linear multiplicative noise for technical reasons. Can one prove such results with more general noise? Second, can one construct the stationary solutions to stochastic Navier-Stokes or Euler equations by convex integration method? Third, analogously to the deterministic case, can one prove the non-uniqueness of solutions to stochastic Navier-Stokes equations at the regularity of $L^2(0, T; H^1(\mathbb{T}^3))$?

(6) Several open problems about stochastic quantization and application of theory of regularity structures to the Yang-Mills problem were suggested. The dimension four, which is of special interest, is the critical dimension and seems to remain out of reach at the time of this workshop.

(7) Various problems concerning mixing and Kuramoto-Sivashinsky equation were suggested. First, can one prove that flows with fixed palenstrophy mix at an exponential rate? Second, can one find an explicit example of an optimal mixing vecor field in $L^\infty(0, T; W^{s,p})$? Finally, can one find a flow that explodes in the H^s norm for almost every initial datum?

(8) It was asked whether one can prove the global existence of strong solutions to the Kuramoto-Sivashinsky equation in dimension strictly higher than one, even starting from small initial data.

In the afternoon sessions, every participant in the workshop, including non-speakers, was actively engaged in discussions and shared more problems of interests. Thereafter, we voted on topics and split into groups of our choice, making significant progress on problems of interest. The special format of the AIM workshops gave us the opportunity to get as much as we can out of a virtual conference. There were extensive interactions among the participants. The conversations have sparkled some truly interesting and creative ideas, which will likely provide bridges between seemingly disconnected topics. We strongly believe that the fruitful discussions we had gave everyone involved a deeper understanding of several fields and will lead to many future collaborations. We also saw much networking building during the workshop, especially for younger researchers. We have no doubt that such networking will benefit us all in the years ahead.

Bibliography

[BV19] T. Buckmaster and V. Vicol, *Nonuniqueness of weak solutions to the Navier-Stokes equation*, Ann. of Math., **189** (2019), 101–144.

[DS09] C. De Lellis and L. Székelyhidi Jr., *The Euler equations as a differential inclusion*, Ann. of Math., **170** (2009), 1417–1436.

[DS10] C. De Lellis and L. Székelyhidi Jr., *On admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal., **195** (2010), 225–260.

[DS13] C. De Lellis and L. Székelyhidi Jr., *Dissipative continuous Euler flows*, Invent. Math., **193** (2013), 377–407.

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