Introduction and Background

Given the complex, uncertain, multi-scale nature of our earth’s climate system it is not surprising that stochastic methods have become increasingly important in the study of Geophysical Fluid Dynamics. This workshop brought together experts in the mathematical community of the equations of liquids and gases and of stochastic differential equations with experts in geosciences who work on numerical simulations of large scale models of the climate and related subsystems to discuss on recent advances in the mathematical and computational theory of the stochastic equations of fluid flow in order to facilitate the cross pollination of ideas with the geophysical community involved with the modeling of the oceans and atmosphere.

Ever since the pioneering work of Bjerknes and Richardson at the turn of the 20th century and culminating in the first modern numerical simulations of the climate by John Von Neumann and his school in the 1950s and 1960s, mathematics and the geosciences have found fruitful interaction through the famous Euler, Navier-Stokes, and advection-diffusion equations and their geophysical counterparts, the Boussinesq and Primitive Equations of the oceans and the atmosphere. Today, these equations form the core of the most advanced General Circulation Models (GCMs) of the atmosphere, the oceans, and for the coupled atmosphere-oceanic system. Probabilistic versions of these equations provide a description of turbulent behavior and may be employed as a means to parameterize unresolved but significant spatial-temporal scales or processes.

Even though these fluids equations have had a long and distinguished history, many fundamental mathematical questions associated with them remain an open challenge, such as the famous Clay Prize problem for the Navier-Stokes equations. In addition, there are many less famous problems which are nevertheless very important and interesting. In particular several new or rapidly developing areas of stochastic analysis, for example the numerics of SPDEs, or the theory of estimation of parameters and states for uncertain processes, need to be further developed or explored in a fluids context.

This workshop brought together a lively mix of specialists in climate modeling and weather prediction alongside experts in the fields of deterministic and stochastic partial differential equations. The week was highlighted by many interesting lectures which promoted significant dialogue and cross-polination of ideas between the two communities. These lectures were followed by lengthy moderated discussions which provided an opportunity for both sides to ask ‘naive’ questions and to formulate novel research problems and directions. This was a rare opportunity for both sides to interact in an informal setting.
**Workshop Focus Areas**

The workshop focused on the following interconnected topics which stand at the forefront of research for both theory and applications

1. **The analytical and stochastic theory of the equations of fluid dynamics and of the atmosphere and the oceans.** While dating back to seminal work in the 1970’s the mathematical theory for the Stochastic Navier-Stokes and Euler Equations continues as an important and rapidly developing field. On the other hand the theory for geophysical equations such as the Boussinesq and Primitive Equations (PEs) and other related equations (the delta-Primitive equation, the quasi-geostrophic equation) have undergone rapid developments in recent years in both the deterministic and stochastic contexts. Although the Primitive equations are technically more involved than the incompressible Navier-Stokes equations, their mathematical theory has very recently been brought to a more advanced level in comparison to the Navier-Stokes equations in terms of the existence, uniqueness, and regularity of solutions in the two-and three-dimensional cases.

2. **Qualitative theory and the long time behavior of solutions.** The study of statistically stationary states for the stochastic Navier-Stokes (and other stochastic equations of fluid flow) provides a model for developed turbulence. The ergodic theory for these systems, an area which has undergone rapid development in recent years, elucidates the relationship between spatial and temporal averages taken over observables. It also provides a means of quantifying the long time behavior of solutions in terms of statistical equalibria. At the pathwise level the theory of random dynamical systems provides a (non-autonomous) description of attractors and holds promise for the study of stability and bifurcation questions.

3. **The numerical analysis of finite and infinite dimensional stochastic evolution equations.** Theory for the numerical analysis of stochastic equations in finite dimensions is now fairly extensive but practical implementation difficulties abound, particularly for higher order schemes. Many delicate issues appear in the stochastic context that do not arise in the deterministic setting such as the type of stochastic evolution (Ito vs. Stratonovich) that a particular scheme will approximate or the accurate generation of stochastic increments. The numerical analysis of stochastic partial differential equations is a newly emerging field with many interesting open questions around weak and strong convergence and stability, particularly for the nonlinear SPDEs of fluid dynamics.

4. **The statistical theory of parameter and state estimation in noisy dynamical systems.** The estimation of partially unknown states or parameters of a noisy system is a central part of the modeling process in the climate community and falls under the general designation of ‘data assimilation’. Dating back to the dramatic successes of the Kalman-Bucy filter in the 1960s these estimation questions have benefited from a rich mathematical literature in the basic theory of stochastic processes. Modern filtering techniques remain partially heuristic and would benefit from being put on a more rigorous mathematical foundation. On the other hand maximum likelihood type parameter estimation techniques, particularly in the infinite dimensional context, is an exciting new direction that has recently been developed in the mathematics community for both linear and non-linear SPDE.
5. **The physical foundations and heuristics for the use of stochastics in geophysical scale models of fluid flow.** The accurate modeling of the earth’s climate system in its great complexity requires the quantification of numerous uncertainties. For example, a full numerical simulation of all the relevant scales of motion for the Primitive Equations (and other geophysical models) remains out of reach for the foreseeable future. Large Eddy Simulation techniques attempt to ameliorate these problems by introducing various ‘closures’ to partially parametrize the unresolved scales of motion. Besides the many physical forms of parameterization stochastic modeling has emerged as an important direction in the field. Indeed some of the effects coming from the unresolved small scales exhibit a persistent random character and are referred to in the physics literature under the designation of ‘stochastic backscatter’.

Lectures

Some highlights from the weeks lectures include the following

- **Joe Tribbia** (NCAR) - Gave a broad overview of the use of stochastics to represent uncertainties in climate data and models.
- **Mohammed Ziane** (USC) - Provided a review of recent advances in the mathematical theory of the Primitive Equations, a basic PDE at the heart of operational climate and weather models. He also discussed mathematical work on a stochastic version of the equations.
- **Boris Rozovskii** (Brown) outlined recent developments in the use of wiener chaos expansions in the numerical and theoretical study of stochastic partial differential equations.
- **Antonio Navarra** (Centro Euro-Mediterraneo sui Cambiamenti Climatici)- Described recent developments in the use of Feynman path integrals to calculate probability distributions for certain stochastic climate models.
- **David Neelin** (UCLA) Discussed approaches to parameter estimation and sensitivity in precipitation models.
- **Mickael Chekroun** (UCLA/Hawaii) Introduced theoretical and practical approaches to Markov approximation of chaotic models. He discussed applications in his joint work with David Neelin.
- **Franco Flandoli** (Pisa) Overviewed the mathematical foundation of the Komogorov equations and discussed practical challenges for the computation of probability distributions for nonlinear Stochastic PDEs.
- **Susan Friedlander** (USC) Discussed recent developments in the understanding of the inertial structure of the 3d Navier-Stokes equations and explained connections to turbulent flows. Motivated by this work she also introduced some novel ‘shell models; which permit the recovery of the fundamental statistical quantities arising in 3D turbulence theory.
- **Armen Shirikyan** (Universite de Cergy-Pontoise, Paris)- Discussed ergodic and mixing properties of the stochastic and randomly kick forced 2d Navier-Stokes equations and related models. He also discussed normal approximation and large deviations for these models.
• **Peter Kloeden** (Goethe-Universitat)- Discussed theoretical and practical issues involved with the numerical simulation of finite and infinite dimensional stochastic equations using Taylor expansion methods.

• **Nathan Glatt-Holtz** (University of Minnesota/Virginia Tech)- Discussed inviscid limits for stochastic fluids equations and relationships with Turbulence theory.

**Moderated Discussions and Break-Out Sessions**

Afternoons were largely devoted to moderated discussions and informal working groups which met consider new interdisciplinary research directions. Highlights of these afternoon sessions included

• Novel approaches to Stochastic Parameterization. **Juan Restrepo** (Arizona) discussed the role of this framework in the context of the dispersion of pollutants in an oceanographic model.

• Approaches to random forcing at the boundaries of various geophysical models were considered.

• Novel ways of introducting of stochasticity into certain cloud formation models were discussed.