Symmetry and convexity in geometric inequalities

organized by
Karoly Boroczky, Galyna Livshyts, and Elisabeth Werner

Workshop Summary

The workshop was dedicated to the study of symmetry in convex geometry. Our aim was to make progress on several conjectures in the field of convexity which involve symmetry assumptions. One of the most important such conjectures is the Log-Brunn-Minkowski conjecture of Boroczky, Lutwak, Yang and Zhang from 2011, which states that for any pair of symmetric convex sets in $\mathbb{R}^n$ and any $\lambda \in [0, 1]$,

$$|\lambda K +_0 (1-\lambda)L| \geq |K|^{\lambda}|L|^{1-\lambda},$$

where $+_0$ stands for the logarithmic addition of convex bodies. It is known from the work of Saroglou that the validity of this conjecture is equivalent to the validity of the same statement for any log-concave measure. Consequently, this conjecture is stronger than the celebrated B-conjecture, which states that the function $\gamma(e^t K)$ is log-concave for any even log-concave measure γ and a symmetric convex set K. Those conjectured inequalities and their equality cases are of central importance in convex geometry, probability and differential geometry.

At the workshop, the participants coming from different fields gathered to learn and discuss this circle of problems. On the first day we had introductory talks by Zhang and X-J Wang, and had a long a productive problem session, during which many interesting and important questions arose. On the second day we had beautiful talks by Kolesnikov and Tatarko, and a very lively discussion was facilitated. Some important methods, such as the Bohner's method and the optimal transport were introduced to the audience. In the afternoon, we split into five groups for focused discussions. On the third day we had very interesting talks by Guan and Zhao, and the workshop participants were excited to learn about the flow approach to the isoperimetric-type inequalities. In the afternoon, the groups decided to remain the same and continued discussions. On the fourth day, Wei and Lu Wang gave exciting talks. The room had the opportunity to learn about very useful and relevant spectral gap estimates for the eigenfunctions of the Laplace operator with homogenous boundary conditions, which is highly relevant to the subject of the conference. Before lunch, X-J Wang explained the continuity method to the participants. In the afternoon, the discussions continued in the same groups. Some participants split their time between several groups, but mostly everyone remained in the same room throughout the workshop. On the fifth day we had a great talk by Shenfeld, who discussed his recent breakthrough result about the equality cases in the Alexandrov-Fenchel inequality. For the rest of the workshop, discussions in groups continued, with a short break for the library tour.

The following discussion groups were formed.

Room 1. The group discussed possible ways of strengthening the celebrated Cafarelli's theorem about optimal transport, with the goal of applying it to the geometric inequalities, such as the B-conjecture and the conjecture of Gardner and Zvavitch. Some progress was made on the initial problem. The group also attempted to verify the conjectured inequality of Cordero and Rotem, but found counterexamples to some weaker forms of the inequality. Later, the group switched to the question of a different strengthening of Cafarelli's theorem, aiming to estimate the Poincare constant of the Gaussian measure restricted to convex sets. Using the ideas of the proof of Cafarelli's theorem and the techniques from ODE, the group managed to find the sharp estimate in dimension 1, with the lower bound attained by linear functions. The group has made the initial computations in the general case, and believed that the ideas similar to the ones in dimension 1 are applicable also in any dimension. The group intends to finish the project after the workshop is done, and to apply the results to obtain improvements in the current estimates for the conjecture of Gardner and Zvavitch, and potentially the multiplicative p-Brunn-Minkowski conjecture.

Room 2. The second group worked on Cordero's conjecture. They verified the conjecture for unconditional sets and for strips. They explored symmetrization method to tackle the conjecture. The group then formulated a reverse conjecture, which they verified for the L_p balls in the plane.

Room 3. The third group worked on the log-concavity of the heat-kernel and spectral gap estimates in the projective space. They managed to make progress in partial cases and intend to continue working on the problem.

Room 4. The fourth group worked on many questions in Harmonic analysis and convexity. They started to study the question about polynomially integrable bodies with respect to the surface areas of sections. They then worked on Busemann and Petty's 8th problem in their famous list. For an origin symmetric convex body in dimension n, the problem relates the area of the section by linear $(n-1)$-plane to the Gauss curvature at the farthest point of the boundary. They could not solve this problem but will continue to work on it. In their studies of Busemann and Petty's 8th problem, the group formulated a very interesting conjecture about the stability of the solution of the Minkowski problem near a ball estimating the L_2 distance of the support function in terms of the L_2-distance of the curvature function. Some insight in this regard was provided to them by two of the members of Room 1.

Room 5. This group was working on possible Brunn-Minkowski type theorems. For the dual Quermassintegrals with index i, it is known to hold if $i = 0$ (the case of volume) or $i = n - 1$. Unfortunately, no progress have been made for the intermediate i even after trying out essentially all known approaches to the Brunn-Minkowski inequality. The group continues to work on the problem.

For L_p surface area measures, the group clarified that neither Brunn-Minkowski type theorem nor its reverse can be expected besides the known classical cases $p = 1$ (surface area) and $p = 0$ (volume).

For the related p-affine surface area in the plane, the group managed to prove the Brunn-Minkowski type theorem if $2/3 \leq p \leq 1$, and found counterexamples indicating that neither Brunn-Minkowski type theorem nor its reverse hold if p is outside of this range. In higher dimensions (dimension at least three), it was known through the work of Colesanti and Ludwig that no Brunn-Minkowski type theorem or its reverse hold even in the case of classical affine surface area (when $p = 1$). The group continues to work on clarifying the other cases of p-affine surface area in higher dimensions.

In addition, some members of Room 1 and Room 2 gathered on the last day to work on the analogue of the Bezout inequality proved by Ivan Soprunov and Artem Zvavitch, trying to show that equality holds only if the main parameter body is a simplex. They hope that the recent method of Shenfeld and Van Handel, settling the equality case of Minkowski second inequality, applies for this question.