Symmetry-breaking of optimal shapes

organized by
Dorin Bucur, Almut Burchard, Richard Laugesen, and Antoine Henrot

Workshop Summary

Overview

This workshop investigated Shape Optimization problems for which symmetry-breaking is believed to occur. The energy functional is radially symmetric and yet its optimizer fails to enjoy full symmetry: the optimal shape is **not** a ball.

The ball does solve many natural shape optimization problems for geometric and physical models, including the classical isoperimetric problem and inequalities for capacity and the first eigenvalue of the Laplacian under Dirichlet, Robin or Neumann boundary conditions. Thus it becomes particularly fascinating to investigate problems where the ball is NOT the best, and to determine the optimal shape in those situations.

The organizers proposed five notable open problems that are believed to involve symmetry-breaking:

(1) Pólya's capacity problem under surface area constraint — double-disk minimizer?
(2) Neumann eigenvalue under perimeter constraint — equilateral triangle and square both maximizers?
(3) geometric moment of inertia inequality — equilateral maximizer?
(4) isodiametric capacity — Riesz exponents for symmetry-holding or breaking?
(5) Alexandrov's area conjecture, given intrinsic diameter — double-disk maximizer?

At the open problem session on the Monday afternoon, participants suggested many additional symmetry-breaking questions. By following AIM's patented and highly effective voting methodology, participants settled on a total of nine problems that were then tackled by working groups during the week. See below for reports from those groups.

Schedule

Monday, June 17.

9:00 Welcome
9:30 Alina Stancu and Evans Harrell
Analytical methods (geometric flows; Riesz means of eigenvalues)
11:15 Edouard Oudet and Nilima Nigam
Numerical methods (interval arithmetic; computer-assisted proofs)
12:30 Lunch
14:30 Plenary problem session
17:00 Happy hour

Tuesday, June 18.

9:00 Almut Burchard, Dorin Bucur
Symmetry-breaking in capacitor problems
11:15 Antoine Henrot
Symmetry breaking in spectral optimization
12:15 Lunch
14:15 Group work
17:00 Happy hour
18:00 Banquet

Wednesday, June 19.

9:00 Krzystof Burdzy
Probabilistic methods example: (the Hot Spots conjecture)
10:45 Iosif Polterovich and Alexandre Girouard
Sharp inequalities for Steklov eigenvalues
14:15 Group work
17:00 Happy hour

Thursday, June 20.

9:00 David Jerison
Geometric flow for the nodal lines of eigenfunctions
11:00 Mickaël Nahon
Spectral problems on the sphere
14:15 Group work
17:00 Happy hour

Friday, June 21.

9:00 Xuefeng Liu
Explicit constants in error formulae in finite element methods
10:15 Carrie Clark, Ilias Ftouhi, Paul Simanjuntak, Ryoki Endo, Chase Reuter, Lukas Bundrock
(Short talks)
14:15 Group work
17:00 Happy hour

Reports from the working groups

First Robin eigenvalue with negative parameter (Bareket conjecture).

Report by: Antoine Henrot
Participants:
Lukas Bundrock, Sebastian Dominguez-Rivera, Antoine Henrot, Nilima Nigam

We looked at the problem

$$
\begin{aligned}
-\Delta u &= \lambda u \quad \text{in } \Omega \\
\frac{\partial u}{\partial n} + \alpha u &= 0 \quad \text{on } \partial\Omega
\end{aligned}
$$

where the Robin parameter α is negative. We are interested in the shape of the set Ω that maximizes the first Robin eigenvalue λ_1 among sets of given area. Our aim was to prove that, in dimension 2, the annulus is the optimal shape for some range of $\alpha \in (-\infty, \alpha^*)$. For that purpose, we wanted to follow a suggestion of D. Krejčiřík. Namely, in his paper with P. Freitas, they were able to compare the first Robin eigenvalue of any domain with the first eigenvalue of a mixed problem on a specific annulus (with Robin boundary condition on the exterior boundary and Neumann boundary condition on the inner boundary). The idea was then to see whether we can find another annulus whose first Robin eigenvalue would be larger than the latter one. A precise analysis of Robin eigenvalues for annuli convinced us that this strategy cannot work, and so the group moved on to other projects.

p-capacity in the plane.

Report by: Dorin Bucur
Participants:
Dorin Bucur, Qiuling Fan, Ilias Ftouhi, Richard Laugesen, Edouard Oudet

In 1947 Pólya conjectured that given a bounded open convex set $\Omega \subset \mathbb{R}^3$, the following inequality occurs:

$$
\frac{\mathrm{Cap}(\Omega)}{\sqrt{|\partial\Omega|}} \geq 4\sqrt{\frac{2}{\pi}},
$$

with equality only if Ω is a flat disk. Above, $|\partial\Omega|$ stands for the surface area of the boundary of Ω and $\mathrm{Cap}(\Omega)$ for the electrostatic capacity of the compact set $\overline{\Omega}$, which is defined as

$$
\mathrm{Cap}(\Omega) = \inf\{\int_{\mathbb{R}^3} |\nabla u|^2 dx : u \in C_0^\infty(\mathbb{R}^3), u \geq 1 \text{ in } \overline{\Omega}\}.
$$

A two dimensional version of the problem, asserting that the logarithmic capacity of a convex planar set is minimized by a double-sided segment under a perimeter constraint ($\mathrm{Cap}_{log}(\Omega)/|\partial\Omega| \geq 1/8$), was conjectured by Pólya and Szegő and proved by Pólya and Schiffer, and later by Pommerenke. This question is handled via conformal mapping.

The group focused on the following question: given $p \in (1,2)$, minimize the p-capacity of a convex planar set under a perimeter constraint. Relying on the Brunn-Minkowksi inequality for the p-capacity and on the linearity of the perimeter under Minkowski addition, we can reduce the problem to the class of triangles. The remaining question is to prove that the triangle has to be degenerate, so a needle, for every $p \in (1,2)$. Some strategies emerged, as for instance using the Serrin type approach of Fragalà and Velichkov in the class of triangles together with some numerical approach in order to give evidence of the structure of the minimizer.

Eigenvalues of systems.

Report by: Antoine Henrot

Participants:
Sebastian Dominguez-Rivera, Alexandre Girouard, Antoine Henrot, David Jerison, Nilima Nigam, Iosif Polterovich

We started by gaining an overview of the different eigenvalue problems for systems, in $\mathbb{R}^2$ and $\mathbb{R}^3$: curl, curl-curl, Maxwell, Stokes, elasticity. Then we chose to focus on the Maxwell system of equations in $\mathbb{R}^3$ with some kind of Dirichlet boundary condition ($U \cdot n = 0$). Thanks to the explicit decomposition of eigenvectors in the case of cylindrical domains, we realized that:

- minimizing the first eigenvalue among domains with a given volume is not interesting (the infimum being zero), but
- minimizing the first eigenvalue among domains with a given surface area seems to be a very interesting problem.

In particular, we conjecture that the minimizing domain for the latter problem is the double-disk. We have been able to prove it in the subclass of cylindrical domains, but of course the general conjecture is much more challenging.

We have developed a sequence of questions (existence of a minimizer in the class of convex domains, continuity of eigenvalues, no strictly convex boundary portions for an optimal domain, ...). The group will work on these different questions during the next months.

Problems with N-gons (Steklov, heat content, Neumann).

Report by: Dorin Bucur
Participants:
Lukas Bundrock, Dorin Bucur, Ryoki Endo, Braxton Osting, Edouard Oudet

Denoting by $\mu_1(\Omega)$ and $\mu_2(\Omega)$ the first two nonzero eigenvalues of the Laplace operator on a planar set $\Omega \subset \mathbb{R}^2$, the question on which the group worked is the following Neumann version of the Pólya-Szegő polygonal Faber-Krahn conjecture: among all quadrilaterals in the plane of prescribed area, prove that the square maximizes $\mu_1(\Omega)$. A natural stronger result, in the sprit of Szegő, would be to prove a similar result for the minimization of $\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)}$ in the same class.

The group focused on this latter question and built a strategy of hybrid type, theoretical-numerical. Precisely, a series of test functions defined on the entire plane have been tested numerically for quadrilaterals lying in a small neighbourhood of the square and, along with some negative answers, supported by numerical evidence we succeeded in finding a couple of them which turned to be relevant. Although a series of steps remain to be done, the knowledge of these valid test functions is very promising towards a rigorous proof. The members of the group remain in contact to complete the proof.

Torsion problem.

Report by: Chris Burdzy
Participants:
Chris Burdzy, Ilias Ftouhi, Chiu-Yen Kao, Xuefeng Liu, Michaël Nahon

The torsion function of a domain represents the expected lifetime of Brownian motion starting from an arbitrary point and killed upon exiting the domain. Among domains with area 1, one would like to find the domain maximizing the gradient of the torsion function at boundary points. An optimal domain is known to exist. Numerical calculations strongly suggest that its boundary contains a line segment.

Over several days, our group developed an outline of a rigorous proof that the boundary must contain a line segment. In addition, we should be able to provide information on other geometric features of the optimizer, such as its inradius. Further, we believe we will be able to develop a numerical approach to the problem that is better than those in existing literature, possibly including rigorous error bounds.

Geometric inequality with moment of inertia.

Report by: Antoine Henrot
Participants:
Phanuel De Andrade Mariano, Evans Harrell, Luis Rademacher, Chase Reuter, Paul Simanjuntak, Alina Stancu

This group studied the purely geometric functional $F(K) = P^2(K)A(K)/I(K)$ where K is a convex set in the plane, $P(K)$ is its perimeter, $A(K)$ its area and $I(K)$ its polar moment of inertia. It is known that a convex maximizer exists that is not the disk, and that the maximizer has to be a polygon. The conjecture, proposed by G. Pólya, is that the maximizer is the equilateral triangle. A nice strategy would be to prove that the ratio $I(K)/A(K)$ satisfies a Brunn–Minkowski inequality. Some numerical simulations done by the participants seem to confirm that this property should hold true. In that case, a classical result would assert that the maximizer has to be indecomposable in the class of convex domains (because the perimeter is linear under Minkowski addition, in 2 dimensions). Therefore, the maximizer would necessarily be a triangle. Moreover, participants proved that among all triangles, the equilateral triangle is maximal. The group intends to go on studying this interesting open question.

Rearrangement problem with dot product.

Report by: Almut Burchard
Participants:
Almut Burchard, Carrie Clark, David Jerison, Chase Reuter, Paul Simanjuntak

A recent inequality of Dongmeng Xi and Yiming Zhao states if $F : \mathbb{R} \to \mathbb{R}$ is convex and even, then

$$\int_K \int_L F(x \cdot y)\, dxdy \geq \int_{K^*} \int_{L^*} F(x \cdot y)\, dxdy$$

for any pair of convex sets $K, L \subset \mathbb{R}^n$. This contains the Blaschke-Santaló inequality for the Mahler product as a special case, and shares its symmetries under linear transformations $x \mapsto Ax$, $y \mapsto A^{-T}y$. We wondered whether the convexity assumptions on the kernel ϕ and the sets K, L are necessary, and considered potential extensions to Gauss space and integrals involving more than two functions. Looking at the proof in detail, we found at its heart a novel averaging argument for n-fold Steiner symmetrizations where the convexity of F plays a crucial role. We consider these new techniques to be highly promising but currently do not see a way forward.

Logarithmic capacity: symmetry breaking in dimensions 3 and 4.

Report by: Almut Burchard
Participants:
Almut Burchard, Carrie Clark, Qiuling Fan, Richard Laugesen, Jie Xiao

It is known that balls maximize the logarithmic capacity among bodies of given diameter in dimension $n = 2$, while symmetry is broken in dimensions $n \geq 5$. It is expected that symmetry is broken also in dimensions 3 and 4. Numerical evidence (from particle-simulations for related aggregation problems) sugest that maximizers may be certain bodies of constant width having tetrahedral symmetry; the equilibrium measures apear to concentrate on the edges of the tetrahedron. We tried to adapt from dimensions $n \geq 5$ to $n = 3, 4$ the existing techniques for creating non-symmetric competitors, as disjoint unions and products of small bodies. We also discussed alternative constructions involving convolutions of equilibrium measures. Despite our efforts, the problem remains open.

Brunn–Minkowski and isodiametric inequality for Riesz capacity.

Report by: Richard Laugesen
Participants:
Almut Burchard, Carrie Clark, Qiuling Fan, Richard Laugesen, Jie Xiao

The classical Brunn–Minkowski inequality says that the n-th root of volume is a concave functional with respect to Minkowski sums of sets in n-dimensional space. The analogous assertion holds for Newtonian capacity, and also for Newtonian capacity of sets in one-lower dimension. In terms of Riesz capacity, that means Brunn–Minkowski is known for exponents $n - 2$ and $n - 1$. The natural conjecture is that should hold for all exponents in $[n - 2, n)$.

In order to attack the conjecture, some "convex convolution of measures" seems to be required, to generate a good test measure that can be inserted into a variational characterization of capacity. It remains unclear what construction might satisfy the constraints in the problem, and so the group refocused on a weaker (but still important!) problem: the isodiametric capacitary conjecture that among convex sets of given diameter, the ball maximizes Riesz capacity. We believe this conjecture can likely be proved by combining the original method of Szegő for the Newtonian case with modern formulas for the fractional Laplacians. The group will continue collaborating on the project in the coming months.