SYZYGIES AND MIRROR SYMMETRY

organized by
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Workshop Summary

Goals of the workshop

The workshop focused on recent breakthroughs in the multigraded commutative algebra of toric varieties coming from homological mirror symmetry.

Mirror symmetry is a deep relationship between symplectic topology and algebraic geometry originally suggested by string theory. The homological mirror symmetry conjecture, proposed by Kontsevich, gives a geometric interpretation of mirror symmetry as an equivalence of categories. The Fukaya category, which captures the Lagrangian intersection theory of a symplectic manifold, is related to the derived category of coherent sheaves on a mirror space. Mirror symmetry has had a profound impact on symplectic topology and algebraic geometry and led to many fruitful interactions between these two fields.

More classically, the field of algebraic geometry is built upon the principle that the geometry of solution sets of polynomial equations can be understood through commutative algebra. This perspective has been thoroughly developed for varieties in projective space where syzygies of the defining ideals of subvarieties have been shown to capture many geometric features of the subvariety. Expressing the geometry in terms of a polynomial ring also allows for computer algebra tools to be brought to bear on computational problems.

Until recently, mirror symmetry has had few direct applications to commutative algebra. However, in early 2023, homological mirror symmetry was used to prove an analog of Hilbert's syzygy theorem for toric varieties conjectured by Berkesch, Erman, and Smith. This provided a key step in developing the geometry of syzygies for toric varieties. The development of this field is made even more interesting due to the nontrivial input of mirror symmetry.

The primary goal of the workshop was to bring together researchers in commutative algebra, algebraic geometry, and symplectic topology to further study the new developments in toric commutative algebra and explore related research directions. As there is little history of interaction between the commutative algebra and the homological mirror symmetry research communities, the workshop led to many new professional relationships and collaborations. The workshop featured talks meant to introduce the main ideas and problems of interest in each field. Working groups were formed during the week on the problems outlined in the next section. We expect the workshop will lead to several new interesting research directions and

hope it will encourage continued communication between commutative algebraists, algebraic geometers, and symplectic topologists.

Problems studied in working groups

Here, we summarize several problems studied by the working groups at the workshop and some of the progress that was made. In each group, a designated contact person volunteered to coordinate future efforts and follow-up.

Comparing resolutions of the diagonal on toric varieties

One of the results that inspired this workshop is a theorem of Hanlon, Hicks, and Lazarev that if Y is a closed toric subvariety of a smooth toric variety X_Σ, then $\mathcal{O}_Y$ admits an explicit resolution by direct sums of line bundles on X_Σ. Applying this result to the diagonal in $X_\Sigma \times X_\Sigma$ implies an analog of Hilbert's Syzygy theorem for toric varieties conjectured by Berkesch, Erman, and Smith. The argument of Hanlon, Hicks, and Lazarev was inspired by mirror symmetry and uses various topological techniques. From this more general perspective, Brown and Erman were able to give a short algebraic proof of the existence of such resolutions. In addition, Favero and Huang presented work in progress giving another topological proof of the existence of these resolutions that directly uses homological mirror symmetry. One of the working groups focused on comparing these three approaches particularly for the diagonal.

The group working on this project began by studying the three approaches and seeing if there was hope for them to be related. They found that the Hanlon-Hicks-Lazarev and Brown-Erman resolutions appeared to be the same (up to homotopy, which is what had been previously proposed). On the other hand, the group discovered examples indicating that the Favero-Huang construction is genuinely distinct and is potentially more closely related to linear resolutions resulting from a Koszul duality perspective.

The group left with a strong outline for how to prove that the Hanlon-Hicks-Lazarev and Brown-Erman resolutions were the same, and they have continued collaborating. A proof of this would open the door to better understanding properties of the resolution and to potentially approaching several open conjectures about resolutions of toric varieties.

Understanding the Cox ring and acyclicity through mirror symmetry

The Cox ring of a toric variety is a multigraded polynomial ring which leads to a connection between homological methods on a toric variety and the study of syzygies over multigraded polynomial rings. Although there is work of Shende exploring the mirror to the Cox construction, homological mirror symmetry is primarily about a connection at the level toric varieties. However, for algebraic purposes, it is necessary to "lift" that connection to the level of Cox rings. This group sought to understand the mirror functor at the level of Cox rings, with a hope of better understanding why the unexpected algebraic acyclicity of the Hanlon-Hicks-Lazarev resolutions.

Some basic steps in understanding how such a lift might be pursued were discussed at the workshop, but this project will require more time and extensive research to develop. Discussions at the workshop made it clear that a detailed mirror dictionary involving the Cox ring and algebraic constructions would be a powerful tool. Some of the ideas that emerged from these group discussions were woven into ongoing work of other groups.

The topology and Morse theory of cellular resolutions
As previously noted, two of the new approaches to toric resolutions rely heavily on using topological properties of the torus. Unbeknownst to some of the symplectic geometers constructing the resolutions, resolutions modeled on a CW complex are called cellular resolutions and have been extensively studied in commutative algebra.

One of the workshop groups createdd a dialogue between symplectic geometers and commutative algebraists on better understanding the role of topology in resolutions. This group also spent some time digesting Favero and Huang's work in progress on resolutions built from the homotopy path algebras that they have introduced. Interesting problems were proposed on using invariants beyond homology of the underlying CW complex to study resolutions and commutative algegbra. The group explored the toric setting where they concluded that all resolutions of interest must come from a decomposition of a real torus where higher homotopy invariants are trivial. This shed further light on toric resolutions and left the door open for more investigations into topological aspects of resolutions in commutative algebra.

Homological mirror symmetry for arithmetic toric varieties
Homological mirror symmetry for toric varieties over algebraically closed fields has been extensively studied and proved using several different techniques. In its most classical form, homological mirror symmetry in the toric setting aims to relate the derived category $D^b Coh(X_\Sigma)$ of a smooth projective toric variety X_Σ to the Fukaya Seidel category $\mathcal{FS}(W_\Sigma)$ of the function $W_\Sigma \colon (\mathbb{C}^*)^n \to \mathbb{C}$ given by

$$W_\Sigma = \sum_{\alpha \in A} z^\alpha$$

where $A \subset \mathbb{Z}^n$ is the set of primitive generators of the fan Σ of X_Σ. Although this is the more concretely stated version of HMS for toric varieties, it only holds when X_Σ is Fano. Other more general versions have been proved using partially wrapped Fukaya categories and/or microlocal sheaf categories.

Over a field which is not algebraically closed, there is an interesting notion of arithmetic toric variety where the algebraic torus acting on the variety is no longer split. These include Severi-Brauer varieties and varieties with no rational points. Arithmetic toric varieties are essentially determined by similar combinatorial data to standard toric varieties and a Galois action on the fan. Recent work, including by some of the workshop participants, has revealed interesting features of derived categories of arithmetic toric varities. Thus, it is natural to explore homological mirror symmetry for arithmetic toric varieties as a starting point for arithmetic applications of homological mirror symmetry.

The group working on this problem discussed small dimensional examples of homological mirror symmetry of the form

$$\mathcal{FS}(W_\Sigma) \simeq D^b Coh(X_\Sigma)$$

for arithmetic toric varieties and the mirror Lagrangians to line bundles. The main idea is to equip the mirror Lagrangians with local systems that record the Galois action. The group will continue to pursue this project and is confident that they can work towards a proof of homological mirror symmetry for arithmetic toric varieties.

Diagonal resolutions and Rouquier dimension of gluings of toric varieties
A central perspective of mirror symmetry going back to its origin in physics is that mirror manifolds should come in degenerating families. This philosophy underlies the Strominger-Yau-Zaslow conjecture and Gross-Siebert program on constructing mirror spaces. It is also at the heart of Seidel and Sheridan's program to prove homological mirror symmetry for Calabi-Yau manifolds. Often, these are toric degenerations and the central fiber is a gluing of toric varieties along toric strata. Thus, from the perspective of mirror symmetry, it is natural to try to generalize the diagonal resolutions and resulting proofs of Orlov's conjecture on Rouquier dimension for toric varieties to gluings of toric varieties along toric stata.

At the workshop, a group worked on understanding Rouquier dimension of the derived categories and diagonal resolutions for these gluings. The group verified that the Rouquier dimension behaved as expected in low dimensional examples as already known due to work of Bai and Côté. It seemed likely a version of these techniques, which rely on homological mirror symmetry, could be used to compute the Rouquier dimension in general for these spaces. Resolutions of the diagonal appeared to be more subtle than expected, but there are hints that a weak version of a diagonal resolution may be possible. Members of this group will continue to pursue this project and its implications.

The mirror to Frobenius and generation
A surprising aspect of the new resolutions of the diagonal on a toric variety X_Σ is that the line bundles that appear are all summands of $(F_\ell)_* \mathcal{O}_{X_\Sigma}$ where F_ℓ is the toric Frobenius morphism. Toric Frobenius is a toric morphism and an integral lift of the Frobenius morphism. The fact that these line bundles appear as summands can be seen through a geometric understanding of toric Frobenius on the mirror. However, a mirror to the absolute Frobenius is not understood. This project was built out of a desire to understand the mirror to Frobenius and to see recent results of Ballard-Iyengar-Lank-Mukhopadhyay-Pollitz on generation in the mirror Fukaya category.

The group worked through a low-dimensional example to see how the Ballard-Iyengar-Lank-Muhkopadhyay-Pollitz result transfers through mirror symmetry. It also became clear that an analogue of the mirror torus Frobenius extends to any symplectic manifold admitting a Lagrangian torus fibration even with some non-compact fibers. The group arrived at a Floer-theoretic proposal for the mirror to the Frobenius morphism. However, they are not yet able to view it as a geometric Lagrangian correspondence and see the role of the characteristic. There are various interesting further directions that will be pursued.

Immaculate line bundles on toric varieties
A line bundle $\mathcal{L}$ on a variety X is called immaculate if

$$H^i(X, \mathcal{L}) = 0 \text{ for all } i.$$

Immaculate line bundles have received significant attention in the toric literature due to their relation with the pursuit of exceptional collections in the derived category. Certain immaculate line bundles also naturally appear in the resolutions that were the focus of this workshop. Although many classes of examples have been studied, most of the general properties of immaculate line bundles on toric varieties remain mysterious. A group at the workshop formed to study the open question of which toric varieties admit infinitely many immaculate line bundles.

In complex dimension two, Wang has given a simple description of all such toric varieties. In higher dimensions, Borisov and Wang have proposed a criterion on X_Σ which they have proved is necessary and sufficient for a special class of 3-dimensional toric varieties. The group re-examined the Borisov-Wang approach and found a possibly new perspective on the problem in terms of matroid theory. There are also hopes that homological mirror symmetry may shed light on this problem as immaculate line bundles correspond to Lagrangian sections with certain vanishing Floer cohomology groups. Members of the group will continue to try to classify toric varieties with infinitely many immaculate line bundles.

Toric BGG correspondence and mirror symmetry
The algebraic Bernstein-Gelfand-Gelfand (BGG) correspondence relates the bounded derived category of a graded polynomial ring S to the bounded derived category of an exterior algebra E. There is also a geometric BGG correspondence that relates the bounded derived category of coherent sheaves on $\mathbb{P}^n$ to the singularity category of E. The BGG correspondences give well understood equivalences of categories that have yielded a large number of applications.

There are multigraded/toric versions of these correspondences as well, but a number of aspects remain less well understood. Since the various derived categories can also be related to certain Fukaya categories from symplectic geometry, this group sought to use that connection to obtain a new perspective on some of the mysterious aspects of the toric BGG correspondence. One key breatkthrough was a "geometric" interpretation of the Tate resolutions of Eisenbud-Fløystad-Schreyer in terms of a sort of infinite Lagrangian surgery operation. This group is continuing to collaborate.