GEOMETRIC TOMOGRAPHY:
Sections of Convex (and Star!) Bodies

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Scope of Geometric Tomography

Discrete tomography

Computerized tomography

Robot vision

Convex geometry

Imaging

Parallel X-rays ↔ Point X-rays

↓

Projections; classical Brunn-Minkowski theory

? ↔

Sections through a fixed point; dual Brunn-Minkowski theory

Integral geometry

Pattern recognition

Minkowski geometry

Stereology and local stereology

Local theory of Banach spaces

X-ray Problems (P. C. Hammer)

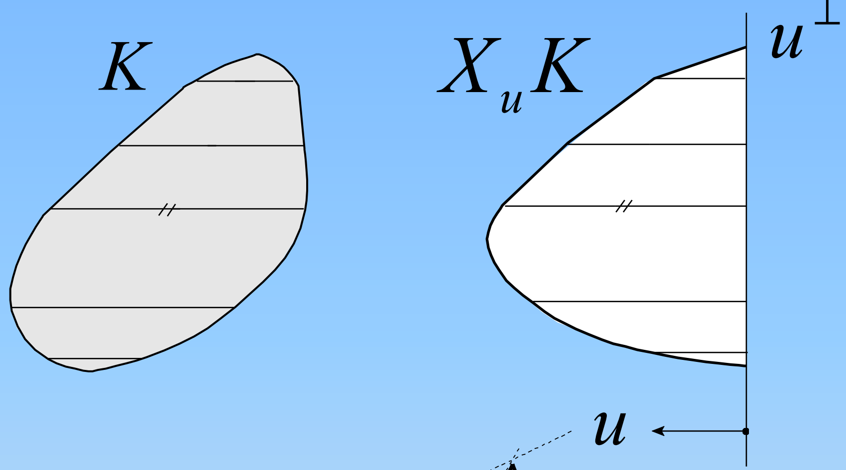
Proc. Symp. Pure Math. Vol VII: Convexity (Providence, RI), AMS, 1963, pp. 498-9

Suppose there is a convex hole in an otherwise homogeneous solid and that X-ray pictures taken are so sharp that the "darkness" at each point determines the length of a chord along an X-ray line. (No diffusion, please.) How many pictures must be taken to permit exact reconstruction of the body if:

a. The X-rays issue from a finite point source?

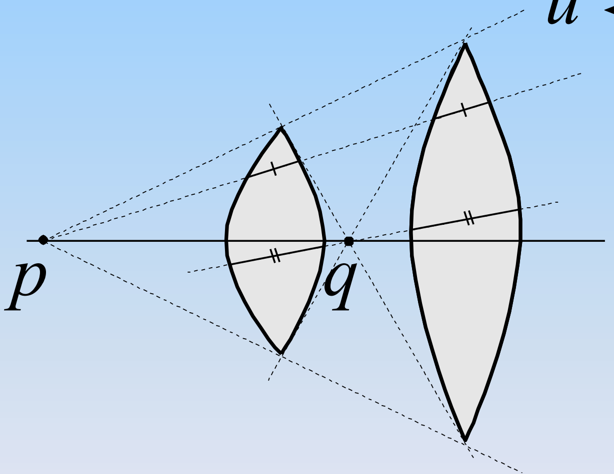
b. The X-rays are assumed parallel?

Parallel and Point X-rays



There are sets of 4 directions so that the parallel X-rays of a planar convex body in those directions determine it uniquely.

(R.J.G. and P. McMullen, 1980)



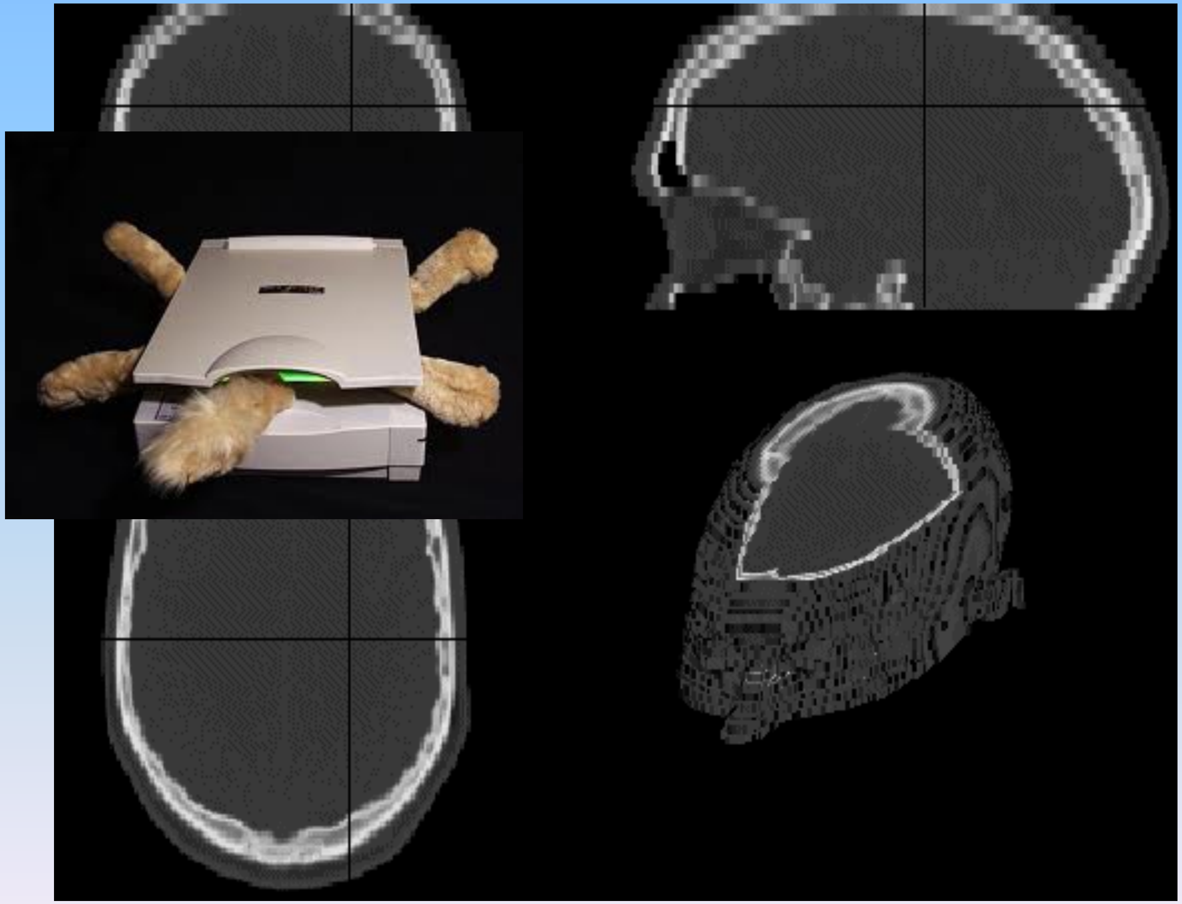
A planar convex body is determined by its X-rays taken from any set of 4 points with no three collinear.

(A. Volčič, 1986)

In these situations a viable algorithm exists for reconstruction, even from noisy measurements.

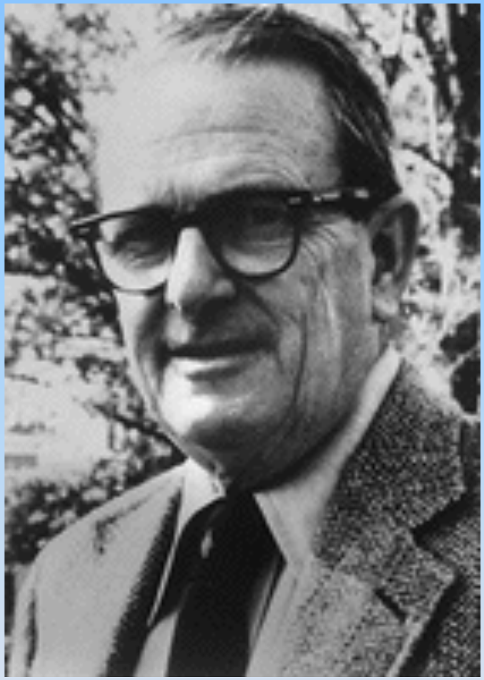
R.J.G. and M. Kiderlen, A solution to Hammer's X-ray reconstruction problem, *Adv. Math.* **214** (2007), 323-343.

Computerized Tomography: CAT Scanner

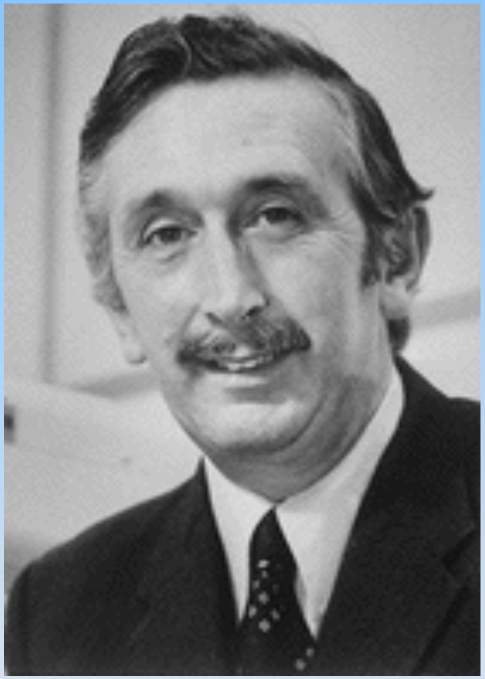


1979 Nobel Prize in Medicine

(Work published in 1963 to 1973)



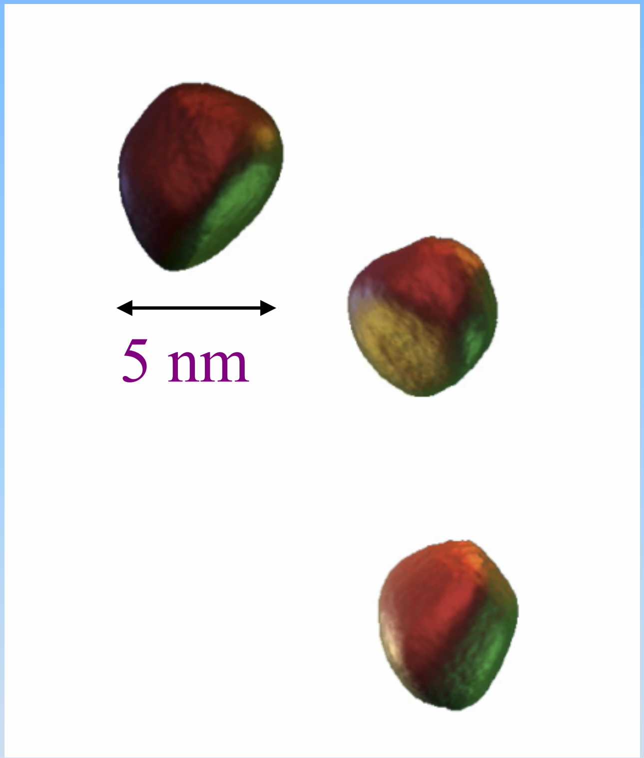
Allan MacLeod Cormack
physicist
(1924 - 1998)



Godfrey Newbold Hounsfield
engineer
(1919 - 2004)

Who Cares?

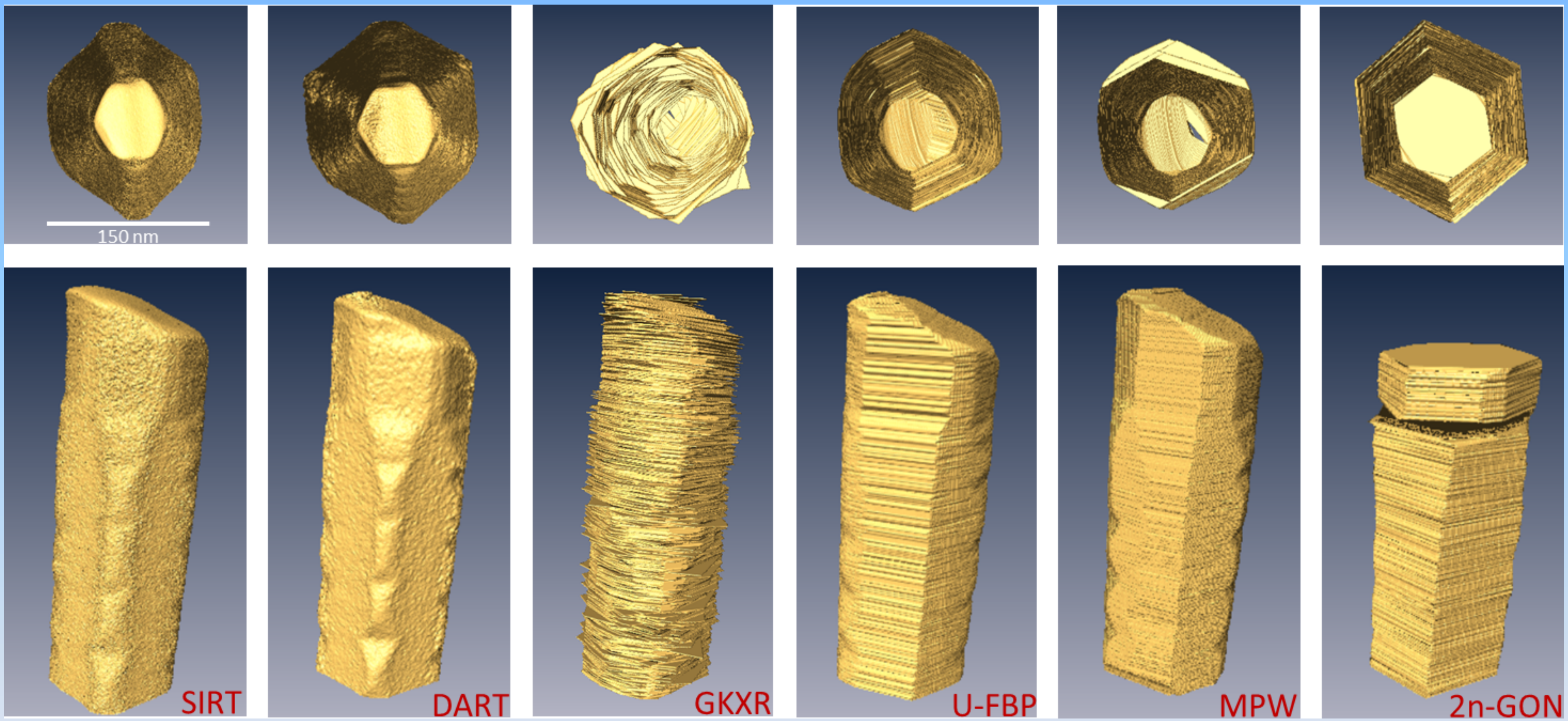
Rafal Dunin-Borkowski, physicist at the Ernst-Ruska-Centre for Microscopy and Spectroscopy with Electrons Institute for Microstructure Research, Jülich, Germany.



5 nanometers = 5×10^{-9} meters

HAADF (High-Angle Annular Dark-Field) electron tomography of platinum catalyst nanoparticles.

Nanowire Reconstructions



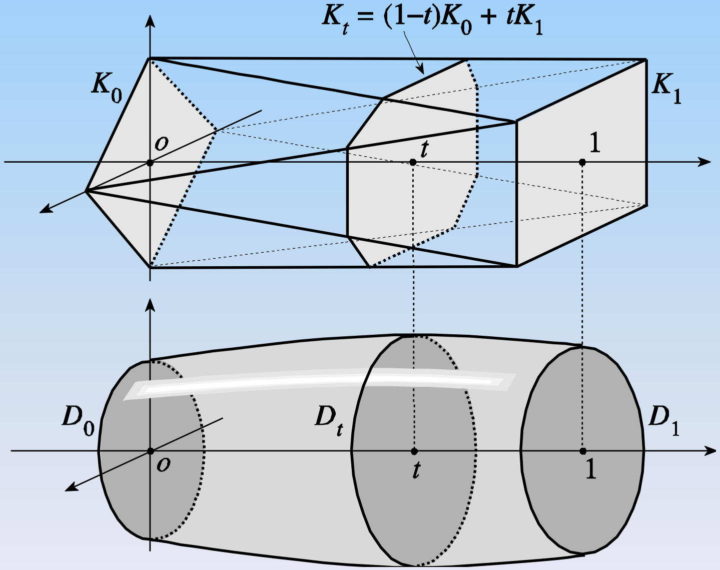
A. Alpers, R.J.G., S. König, R. S. Pennington, C. B. Boothroyd, L. Houben, R. Dunin-Borkowski, and K. J. Batenburg, Geometric reconstruction methods in electron tomography, *Ultramicroscopy* **128** (2013), 42-54.

Unsolved Problems I

"Geometric Tomography," second edition, poses 66 open problems.

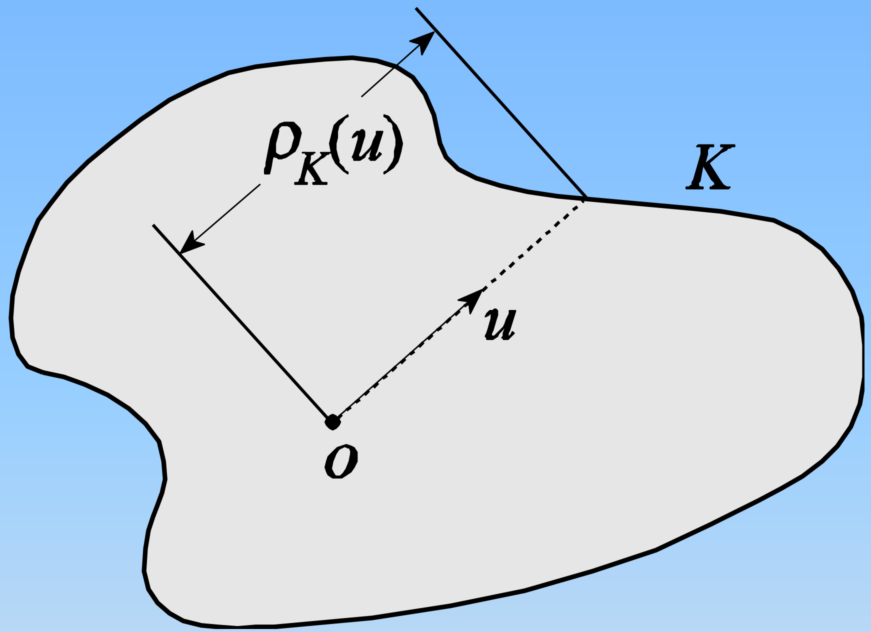
Some examples of problems on X-rays still open:

- Is a convex body in R^3 determined by its parallel X-rays in any set of 7 directions with no three coplanar?
- Are there finite sets of directions in R^3 such that a convex body is determined by its 2-dimensional X-rays orthogonal to these directions?



- Can planar convex bodies be successively determined by their parallel X-rays?

The Radial Function and Star Bodies



Star body: Body star-shaped at *o* whose radial function is positive and continuous,

OR one of several other alternative definitions in the literature!

E. Lutwak, Dual mixed volumes, *Pacific J. Math.* **58** (1975), 531-538.

Polarity Usually Fails

Theorem 1. (Süss, Nakajima, 1932.) Suppose that $2 \le k \le n-1$ and that K and L are compact convex sets in $\mathbb{R}^n$. If all k-projections $K|S$ and $L|S$ of K and L are homothetic (or translates), then K and L are homothetic (or translates, respectively).

Theorem 1'. (Rogers, 1965.) Suppose that $2 \le k \le n-1$ and that K and L are compact convex sets in $\mathbb{R}^n$ containing the origin in their relative interiors. If all k-sections $K \cap S$ and $L \cap S$ of K and L are homothetic (or translates), then K and L are homothetic (or translates, respectively).

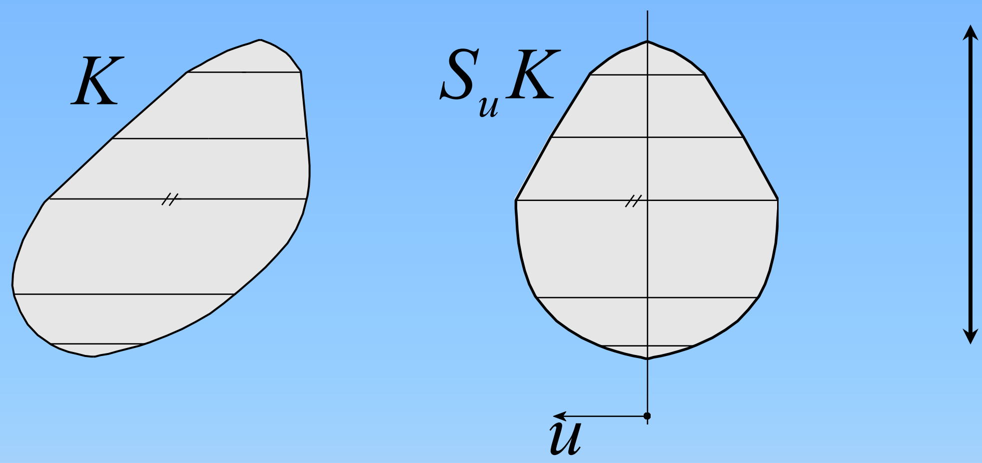
~~$K^o \cap S = (K|S)^o$~~

Open Problem. (G, Problem 7.1.) Does Theorem 1' hold for star bodies, or perhaps more generally still?

Lutwak's Dictionary

Convex bodies	Star bodies
Projections	Sections through o
Support function h_K	Radial function ρ_K
Brightness function b_K	Section function s_K
Projection body ΠK	Intersection body IK
Cosine transform	Spherical Radon transform
Surface area measure	ρ_K^{n-1}
Mixed volumes	Dual mixed volumes
Brunn-Minkowski ineq.	Dual B-M inequality
Aleksandrov-Fenchel	Dual A-F inequality

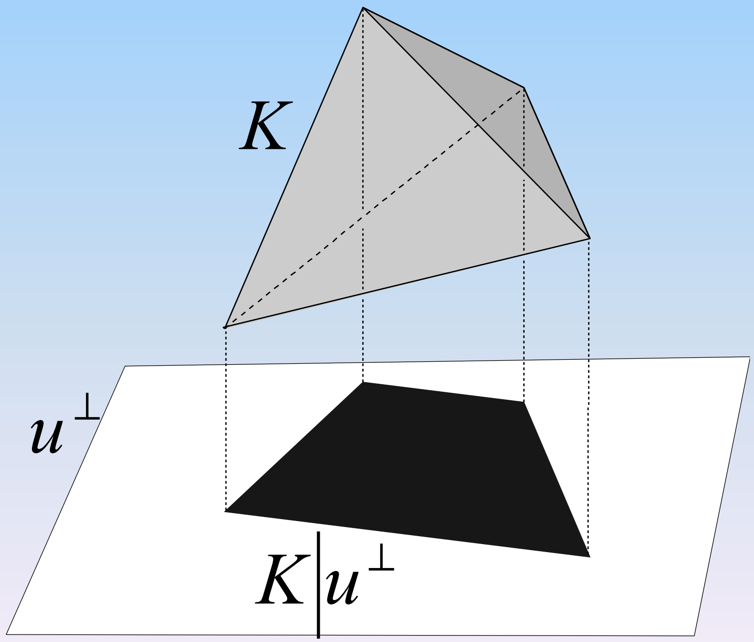
Width and Brightness



width function

$$w_K(u) = V(K|l_u)$$

$$= h_K(u) + h_K(-u)$$



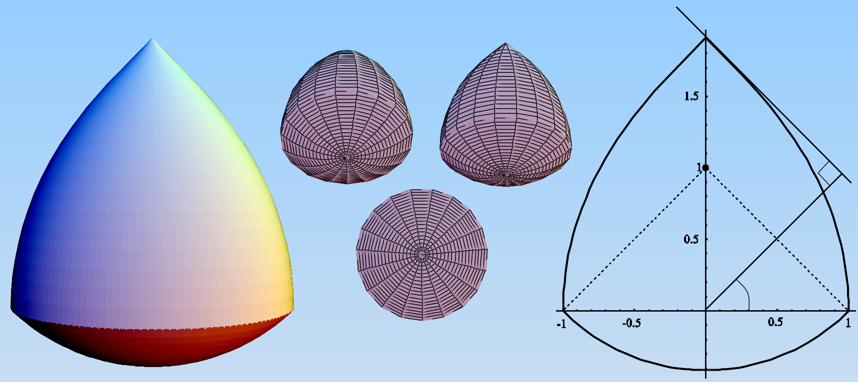
brightness function

$$b_K(u) = V(K|u^\perp)$$

Aleksandrov's Projection Theorem

For *o-symmetric* convex bodies K and L,

$$b_K(u) = b_L(u) \ \forall u \ \Rightarrow \ K = L.$$

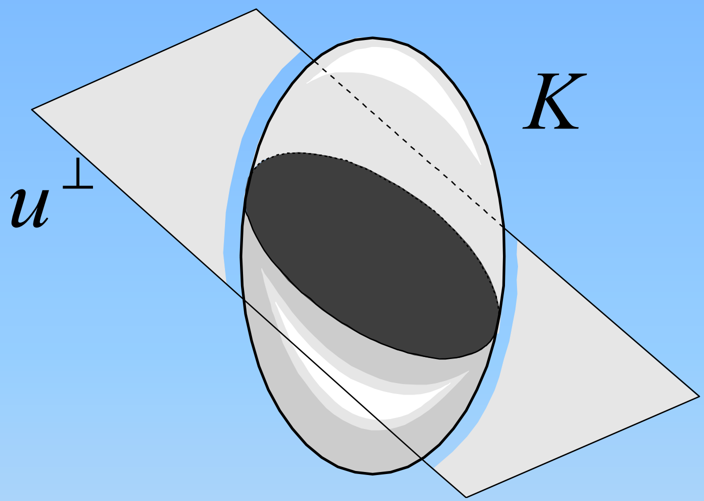


$$b_K(u) = \frac{1}{2}\int_{S^{n-1}} |u \cdot v| \, dS(K, v)$$

← Cauchy's projection formula

Cosine transform of surface area measure of K

Funk-Minkowski Section Theorem



section function

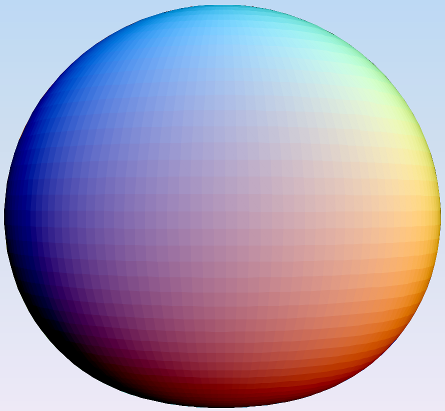
$$s_K(u) = V(K \cap u^\perp)$$
$$= \frac{1}{n-1} \int_{S^{n-1} \cap u^\perp} \rho_K(v)^{n-1}\, dv$$

Spherical Radon transform of (n-1)st power of radial function of K

Funk-Minkowski section theorem:

For *o-symmetric* star bodies K and L,

$$s_K(u) = s_L(u)\ \forall u \Rightarrow K = L.$$



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Is Lutwak's Dictionary Infallible?

Theorem 2. (Goodey, Schneider, and Weil,1997.) Most convex bodies in $\mathbb{R}^n$ are determined, among all convex bodies, up to translation and reflection in *o*, by their width and brightness functions.

Theorem 2'. (R.J.G., Soranzo, and Volčič, 1999.) The set of star bodies in $\mathbb{R}^n$ that are determined, among all star bodies, up to reflection in *o*, by their *k*-section functions for all *k*, is nowhere dense.

Notice that this (very rare) phenomenon does not apply within the class of *o*-symmetric bodies. There I have no example of this sort.

Two Unnatural Problems

1. (Busemann, Petty, 1956.) If K and L are o-symmetric convex bodies in $\mathbb{R}^n$ such that $s_K(u) \le s_L(u)$ for all u in S^{n-1}, is $V(K) \le V(L)$?

Solved: Yes, $n = 2$ (trivial), $n = 3$ (R.J.G., 1994), $n = 4$ (Zhang,1999).

No, $n \ge 5$ (Papadimitrakis, 1992, R.J.G., Zhang, 1994).

Unified: R.J.G., Koldobsky, and Schlumprecht, 1999.

(Aleksandrov, 1937.) If K and L are o-symmetric convex bodies in $\mathbb{R}^3$ whose projections onto every plane have equal perimeters, then $K = L$.

2. ***Open Problem***. (R.J.G., 1995; G, Problem 7.6.) If K and L are o-symmetric star bodies in $\mathbb{R}^3$ whose sections by every plane through o have equal ~~perimeters~~, is $K = L$?

Yes. R.J.G. and Volčič, 1994.

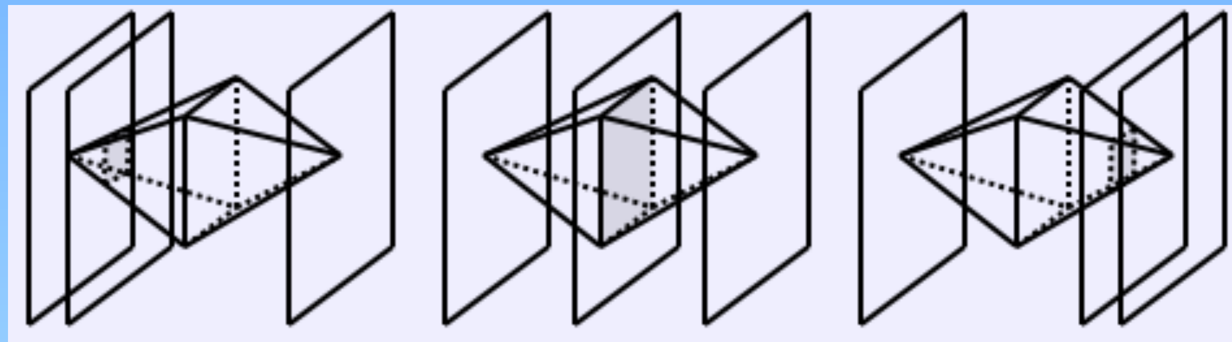
Yes for convex polytopes. Yaskin, 2010.

No without "o-symmetric", even for convex. Ryabogin and Yaskin, 2013.

$$\tilde{V}(K \cap S, B \cap S) = \frac{1}{2} \int_{S^2 \cap S} \rho_K(u)\,du$$

Maximal Section Function

Suppose that $n \geq 3$.



$$m_K(u) = \max_{-\infty < t < \infty} V_{n-1}(K \cap (u^{\perp} + tu))$$

Open Problem. (Bonnesen, 1926.) Do the brightness function and maximal section function together determine every convex body (up to translation and reflection in the origin)? Even open for balls.

(Problem of Klee, 1969.) The maximal section function alone does not suffice…: R.J.G., D. Ryabogin, V. Yaskin, and A. Zvavitch, *J. Differential Geom.* **91** (2012), 261-279.

(Problem of Klee, 1969.) …even for balls! : F. Nazarov, D. Ryabogin, and A. Zvavitch, *J. Amer. Math. Soc.*, to appear.

Discrete versions (R.J.G., P. Gronchi, and C. Zong, 2005) are still open!

Affinely Equivalent Sections

Theorem. (Süss, 1925; Schneider, 1980.) Suppose that $2 \leq k \leq n-1$ and that K is a star body in $\mathbb{R}^n$ with o in int K. If all k-sections $K \cap S$ are congruent, then K is an o-symmetric ball.

Open Problem. (Banach, ????; G, Problem 7.4.) Suppose that $2 \leq k \leq n-1$ and that K is a star body in $\mathbb{R}^n$. If all k-sections $K \cap S$ are affinely equivalent, is K an ellipsoid?

Montejano's result on complete turnings, 1991 → K is either an ellipsoid or o-symmetric.

A *complete turning* of an $(n-1)$-dimensional compact convex set C in $\mathbb{R}^n$ is an even function $C(u)$ on S^{n-1}, continuous in the Hausdorff metric, such that $C(u) \subset u^{\perp}$ is congruent to C.

Gromov, 1967: Yes for K convex and either $k \leq n-2$ or n is odd and $k = n-1$.

Unsolved for $n = 4$ and $k = 3$.

Busemann Intersection Inequality

If K is a convex body in $\boldsymbol{R}^n$ containing the origin in its interior, then

$$V(IK) \le \frac{\kappa_{n-1}^{n}}{\kappa_{n}^{n-2}} V(K)^{n-1},$$

with equality if and only if K is an o-symmetric ellipsoid.

H. Busemann, Volume in terms of concurrent cross-sections, *Pacific J. Math.* **3** (1953), 1-12.

$$V(IK) = \frac{1}{n} \int_{S^{n-1}} V(K \cap u^{\perp})^n \, du$$

Generalized Busemann Intersection Inequality

If K is a bounded Borel set in $\boldsymbol{R}^n$ and $1 \le i \le n$, then

$$\int_{G(n,i)} V_i(K \cap S)^n \, dS \le \frac{\kappa_i^n}{\kappa_n^i} V_n(K)^i,$$

with equality when $1 < i < n$ if and only if K is an o-symmetric ellipsoid and when $i = 1$ if and only if K is an o-symmetric star body, modulo sets of measure zero.

H. Busemann and E. Straus, 1960; E. Grinberg, 1991; R. E. Pfiefer, 1990; R.J.G., E. Vedel Jensen, and A. Volčič, 2003.

R.J.G., The dual Brunn-Minkowski theory for bounded Borel sets: Dual affine quermassintegrals and inequalities, *Adv. Math.* **216** (2007), 358-386.

*p*th Radial Addition

For $-\infty \leq p \neq 0 \leq \infty$, $x \in \mathbb{R}^n \setminus \{o\}$, and $K, L \in \mathcal{S}^n$, let

$$\rho_{K \tilde{+}_p L}(x)^p = \rho_K(x)^p + \rho_L(x)^p,$$

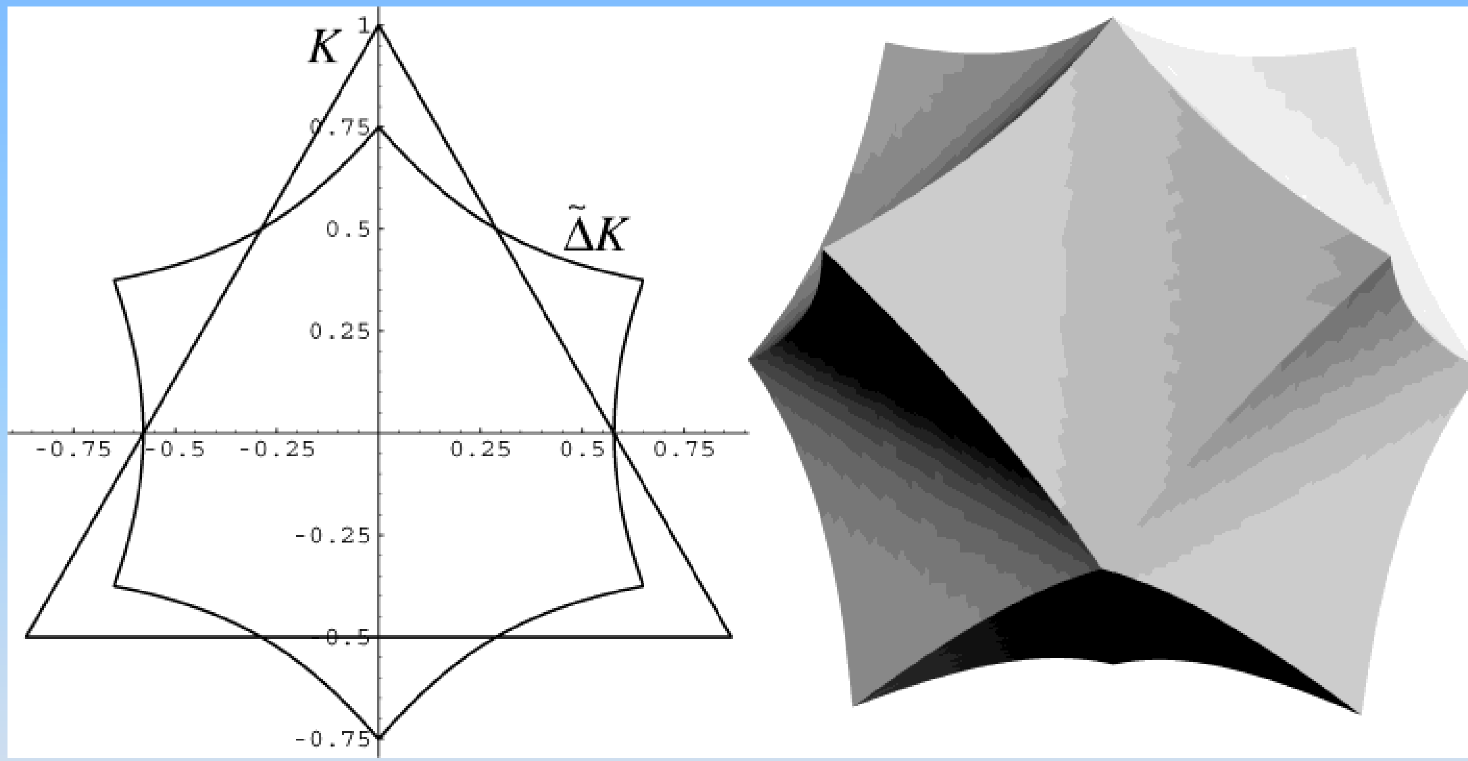
if $p > 0$ or $p < 0$ and $\rho_K(x),\ \rho_L(x) > 0$, and

$$\rho_{K \tilde{+}_p L}(x) = 0,$$

otherwise. Note that

$$K \tilde{+}_{-\infty} L = K \cap L \text{ and } K \tilde{+}_{\infty} L = K \cup L.$$

Chordal Symmetral



$$\tilde{\Delta}K = \frac{1}{2}\left(K \tilde{+} (-K)\right)$$

Characterization of *p*th Radial Addition

Operations $*:(\mathcal{S}_s^n)^2 \to \mathcal{S}^n$ that are associative, continuous in the radial metric, homogeneous of degree 1, and rotation and section covariant are precisely those defined for all $K, L \in \mathcal{S}_s^n$ by

$$K * L = \{o\},\ K * L = K,\ \ K * L = L, \text{ or}$$

$$* = \widetilde{+}_p,\ \ -\infty \le p \le \infty,\ \ p \ne 0.$$

Various examples show that none of the assumptions can be omitted.

R.J.G., D. Hug, and W. Weil, Operations between sets in geometry, *J. Europ. Math. Soc.*, to appear.

Open Problem. Find a nice characterization of *p*th radial addition (or just radial addition) for operations $*:(\mathcal{S}^n)^2 \to \mathcal{S}^n$.

Some Properties

Commutativity: $K * L = L * K.$

Associativity: $K * (L * M) = (K * L) * M.$

Homogeneity of degree *k*: $(rK) * (rL) = r^k (K * L),\ r \geq 0.$

Identity: $K * \{o\} = K = \{o\} * K.$

Continuity: $K_m \to K, L_m \to L \Rightarrow K_m * L_m \to K * L.$

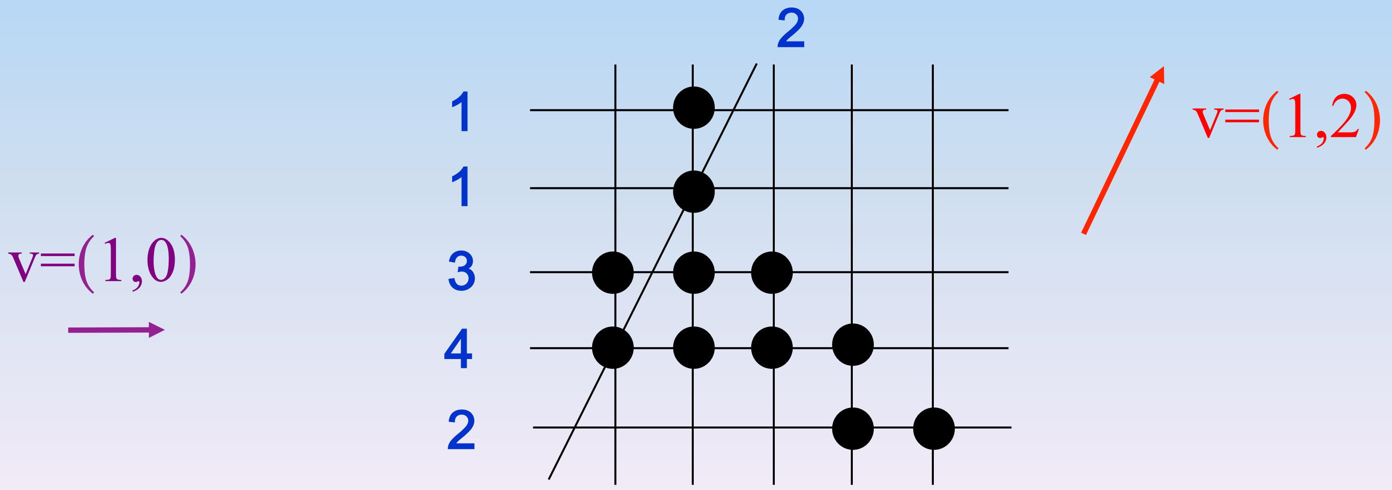
GL(n) covariance: $\phi(K * L) = \phi(K) * \phi(L),\ \phi \in GL(n).$

Section covariance: $(K * L) \cap S = (K \cap S) * (L \cap S).$

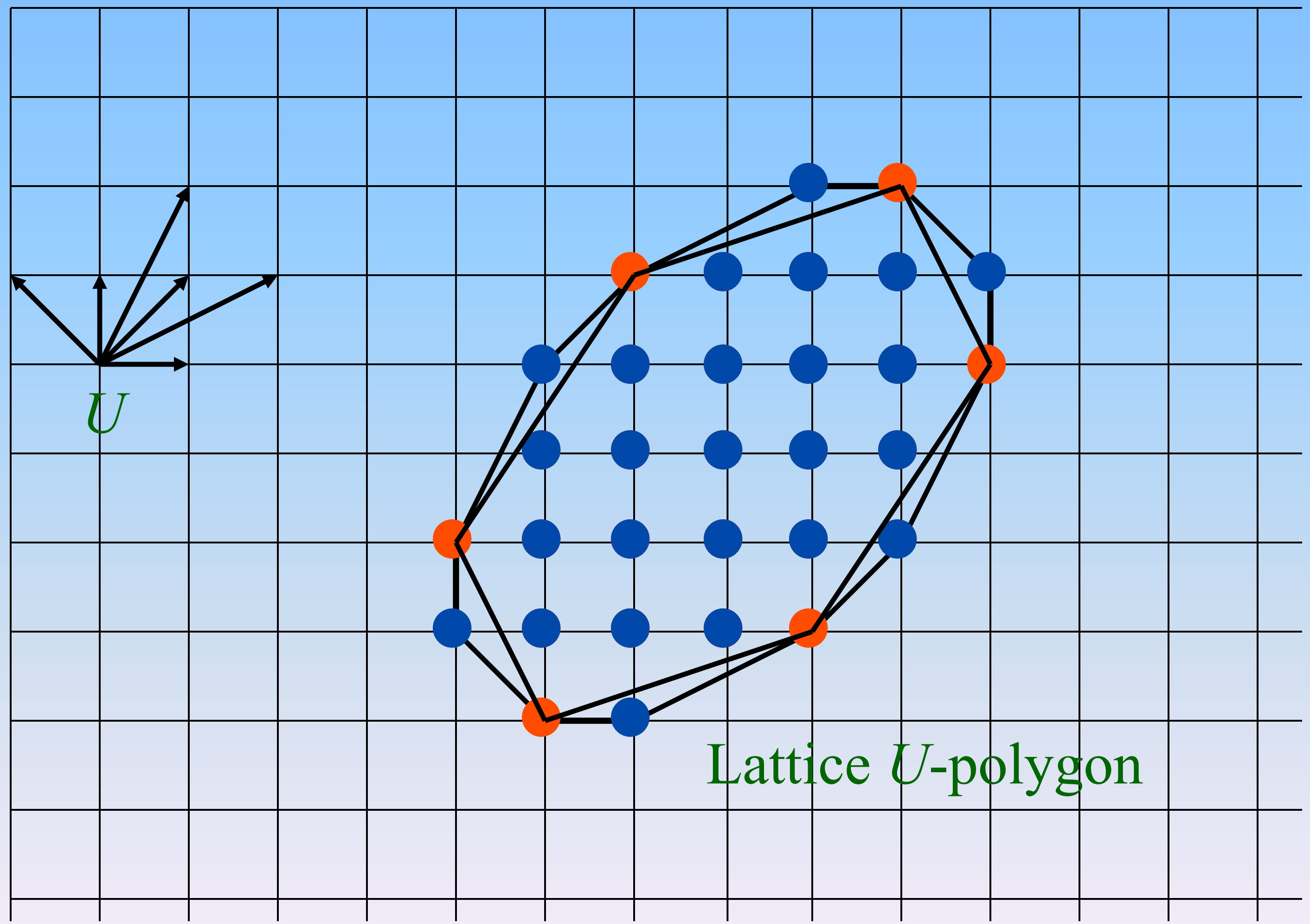
$\Diamond : \mathscr{K}^n \to \mathscr{K}_s^n$ or $\Diamond : \mathscr{S}^n \to \mathscr{S}_s^n \ldots$

Discrete Parallel X-rays

- *F* is a lattice set, a finite subset of the integer lattice.
- *v* is a lattice direction, a vector with integer coordinates.
- $X_v F$ is the (discrete) parallel X-ray of *F* in the direction *v*, giving the number of points in *F* lying on each line parallel to *v*. (Line sums.)



Convex Lattice Sets



Uniqueness For Convex Lattice Sets

A set of m lattice directions is such that the discrete parallel X-rays in these directions determine convex lattice sets if:

- $m = 3$, never.
- $m > 6$, always.
- $m = 4,5,6$, if and only if the cross ratio of the slopes of any 4 directions is NOT 3/2, 2, 4/3, 3, or 4.

$$\langle x_1, x_2, x_3, x_4 \rangle = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_4 - x_1)(x_3 - x_2)}$$

Proof uses algebraic tools: p-adic valuations.

R.J.G. and P. Gritzmann, Discrete tomography: Determination of finite sets by X-rays, *Trans. Amer. Math. Soc.* **349** (1997), 2271-2296.

S. Brunetti and A. Daurat, An algorithm reconstructing lattice convex sets, *Theor. Comp. Sci.* **304** (2003), 35-57.