‘Convexity’ of Intersection Bodies

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The radial function $\rho_K$ of a body $K$ is defined by

$$\rho_K(u) = \max \{ \lambda \geq 0 : \lambda u \in K \}, \quad u \in S^{n-1}.$$ 

Note also that

$$\rho_K(u) = \|u\|_K^{-1}$$ 

for $u \in S^{n-1}$, where $\|\cdot\|_K$ is the Minkowski functional of $K$.

A star body is a body whose radial function is positive and continuous.
In 1988, Lutwak introduced the notion of intersection bodies.

Let $K$ be a star body in $\mathbb{R}^n$. The intersection body of $K$, denoted by $IK$, is a body whose radial function is

$$
\rho_{IK}(u) = |K \cap u^\perp| \quad \text{for each } u \in S^{n-1}.
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$$\rho_{IK}(u) = |K \cap u^\perp| \quad \text{for each } u \in S^{n-1}.$$
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$B_2^n$ $IB_2^n$
Intersection bodies: Examples

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- For $K \subset \mathbb{R}^2$ symmetric, $IK$ is a rotation of $2K$ by angle $\pi/2$.
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- For $K \subset \mathbb{R}^2$ symmetric, $IK$ is a rotation of $2K$ by angle $\pi/2$.

- For $T \in \text{GL}(n)$,

$$I(TK) = |\det T| (T^{-1})^*(IK).$$
Related to the solution for the **Busemann-Petty problem**: For symmetric convex bodies $K, L$ in $\mathbb{R}^n$, is it true that

$$|K \cap H| \geq |L \cap H| \quad \text{for all hyperplanes } H \quad \Rightarrow \quad |K| \geq |L|?$$

[yes if $n \leq 4$, no if $n \geq 5$]. Larman, Rogers, Ball, Lutwak, Gardner, Zhang, Koldobsky, Schlumprecht.
Intersection bodies: motivations

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- Connection with the **Spherical Radon transform** $\mathcal{R}$.

$$\rho_{IK}(\theta) = |K \cap \theta^\perp| = \int_{S^{n-1} \cap \theta^\perp} \int_0^{\rho_K(u)} r^{n-2} dr du$$

$$= \frac{1}{n-1} \int_{S^{n-1} \cap \theta} \rho_K^{n-1}(u) du = \frac{1}{n-1} \mathcal{R} \rho_K^{n-1}(\theta),$$

that is, $\rho_{IK} = \frac{1}{n-1} \mathcal{R} \rho_K^{n-1}$. 

Jaegil Kim  |  Convexity of Intersection Bodies
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- "Convexity" is preserved under the intersection body operator.
**Busemann’s Theorem**

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- "Convexity" is preserved under the intersection body operator.

**Main Question:** How much of ‘convexity’ is preserved under the intersection body operator $I$?
(1) **Quasi-convexity**

$K$ is $q$-convex if $t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K$ whenever $x, y \in K$, $t \in [0, 1]$

"convexity" increases as $q \to 1$. 

"convexity" increases as $d_{BM}(K, B_n^2) \to 1$.

"convexity" increases as $\delta_K(\varepsilon)$ does.
How to measure ‘convexity’

(1) Quasi-convexity

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"convexity" increases as $q \to 1$.

(2) Banach-Mazur distance from $B_2^n$

$$d_{BM}(K, B_2^n) = \min \left\{ r : B_2^n \subset TK \subset rB_2^n, T \in GL(n) \right\}$$

"convexity" increases as $d_{BM}(K, B_2^n) \to 1$. 
(1) **Quasi-convexity**

A set $K$ is $q$-convex if $t^{rac{1}{q}}x + (1 - t)^{rac{1}{q}}y \in K$ whenever $x, y \in K$, $t \in [0, 1]$. The "convexity" increases as $q \to 1$.

(2) **Banach-Mazur distance from $B_2^n$**

$$d_{BM}(K, B_2^n) = \min \left\{ r : B_2^n \subset TK \subset rB_2^n, T \in GL(n) \right\}$$

The "convexity" increases as $d_{BM}(K, B_2^n) \to 1$.

(3) **Modulus of convexity**

$$\delta_K(\varepsilon) = \min \left\{ 1 - \frac{1}{2} \|x + y\|_K : x, y \in K, \|x - y\|_K \geq \varepsilon \right\}$$

The "convexity" increases as $\delta_K(\varepsilon)$ does.
Let $0 < q \leq 1$. A star body $K \subset \mathbb{R}^n$ is called $q$-convex if

$$t^{\frac{1}{q}} x + (1 - t)^{\frac{1}{q}} y \in K \quad \text{whenever } x, y \in K, t \in [0, 1]$$

or, equivalently, $\|x + y\|^q_K \leq \|x\|^q_K + \|y\|^q_K$
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$$B^2_q = \{(x, y) : |x|^q + |y|^q \leq 1\}$$

$q = 1/2$
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$$B_q^2 = \{(x, y) : |x|^q + |y|^q \leq 1\}$$

$q = 1/8$  
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$q = 1$ (convex)
Let $0 < q \leq 1$. A star body $K \subset \mathbb{R}^n$ is called $q$-convex if

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\[ q = 1/8 \quad q = 1/4 \quad q = 1/2 \quad q = 3/4 \quad q = 1 \, \text{(convex)} \]

**Question** If $K$ is $q$-convex, for which $q'$ the intersection body $IK$ is $q'$-convex?
Theorem [K.-Yaskin-Zvavitch, 2011]
Let $K$ be a symmetric star body in $\mathbb{R}^n$ and $0 < q \leq 1$. Then, if $K$ is $q$-convex, $IK$ is $q'$-convex where

$$q' = [(1/q - 1)(n-1) + 1]^{-1}$$

- If $q = 1$, then $q' = 1$: (just Busemann’s theorem!)
- $q' \leq q$, but the formula for $q'$ is sharp asymptotically.
Theorem [K.-Yaskin-Zvavitch, 2011]
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More generally, for a log-concave measure $\mu$ on $\mathbb{R}^n$,

Consider the intersection body $I_\mu K \text{ w.r.t. } \mu$ defined by

$$\rho_{IK}(u) = \mu(K \cap u^\perp) \quad \forall u \in S^{n-1}.$$ 

Then the above theorem still holds for $I_\mu K$. 
Quasi-convexity: Example

Let \( K = \left\{ t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y : x \in C, y \in -C, 0 \leq t \leq 1 \right\} \),
where \( C = \{(1, x_2, \ldots, x_n) : |x_2|, \ldots, |x_n| \leq 1\} \).

\( K \) is \( q \)-convex.
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- $K$ is $q$-convex.

$$|K \cap (e_1 \pm e_2)^\perp| \geq 2^{n-1/2}$$
Let $K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y : x \in C, y \in -C, 0 \leq t \leq 1 \right\}$, where $C = \{(1, x_2, \ldots, x_n) : |x_2|, \ldots, |x_n| \leq 1\}$.

- $K$ is $q$-convex.

- $\|e_1\|_{IK} = 2^{(1/q-2)(n-1)}$, $\|e_1 \pm e_2\|_{IK} \lesssim 2^{1-n}$.
Let $K = \left\{ t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y : x \in C, y \in -C, 0 \leq t \leq 1 \right\}$, where $C = \{(1, x_2, \ldots, x_n) : |x_2|, \ldots, |x_n| \leq 1\}$.

- $K$ is $q$-convex.
- $\|e_1\|_{IK} = 2^{(1/q-2)(n-1)}$, $\|e_1 + e_2\|_{IK} \lesssim 2^{1-n}$.
- If $IK$ is $q'$-convex, then
  \[
  \|2e_1\|_{IK}^{q'} \leq \|e_1 + e_2\|_{IK}^{q'} + \|e_1 - e_2\|_{IK}^{q'}
  \]
  gives
  \[
  q' \lesssim [(1/q - 1)(n-1) + 1]^{-1}.
  \]
(2) Banach-Mazur distance from \( B_2^n \)

[Hensley, 1980] For any symmetric convex body \( K \) in \( \mathbb{R}^n \),

\[
d_{BM}(IK, B_2^n) \leq C
\]

where \( C > 1 \) is a universal constant.

- Note that \( d_{BM}(B_\infty^n, B_2^n) = \sqrt{n} \). In fact, a lot of symmetric convex bodies are very far from the ball in high dimension.

- Nevertheless, the above theorem says that their intersection bodies should be bounded from the ball by an absolute constant.
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- Nevertheless, the above theorem says that their intersection bodies should be bounded from the ball by an absolute constant.

Questions [Lutwak, 1990’s]

Are the followings true?

- $d_{BM}(IK, B_2^n) \leq d_{BM}(K, B_2^n)$,
- $d_{BM}(IK, K) = 1 \Rightarrow d_{BM}(K, B_2^n) = 1$,
- $d_{BM}(I^m K, B_2^n) \to 1$ as $m \to \infty$.
[Fish-Nazarov-Ryabogin-Zvavitch, 2011] If a star body $K$ is close enough to the ball $B^n_2$, then

$$d_{BM}(I^m K, B^n_2) \to 1 \quad \text{as} \quad m \to \infty.$$
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Theorem [Alfonseca-K., 2013] For every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for every $n \geq N$ and any convex body $K$ of revolution in $\mathbb{R}^n$,

$$d_{BM}(I^2 K, B_2^n) \leq 1 + \varepsilon.$$

- Roughly speaking, the double intersection body of a body of revolution is close enough to an ellipsoid in high dimension.
- But the single intersection body is not enough for the above theorem.
Negative answer for star bodies

\[ d_{BM}(IK, B_2^n) \leq d_{BM}(K, B_2^n) \] may not be true without convexity

Consider \( K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y : x \in C, y \in -C, 0 \leq t \leq 1 \right\} \),

\[ K \]
Negative answer for star bodies

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\[ 2^{1-1/q}B_2^n \subset K \]
Negative answer for star bodies

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\[ d_{BM}(K, B_2^n) \leq 2^{1/q-1} \sqrt{n}. \]
Negative answer for star bodies

\[ d_{BM}(IK, B^n_2) \leq d_{BM}(K, B^n_2) \] may not be true without convexity

Consider \( K = \left\{ t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y : x \in C, y \in -C, 0 \leq t \leq 1 \right\} \),

- \( d_{BM}(K, B^n_2) \leq 2^{1/q-1} \sqrt{n} \).
- Note that \( IK \) is \( q' \)-convex where \( q' = [(1/q-1)(n-1) + 1]^{-1} \).

So, \( d_{BM}(IK, B^n_2) \geq 2^{(1/q-1)(n-1)} \).
- \( d_{BM}(IK, B^n_2) \gg d_{BM}(K, B^n_2) \).
(3) Modulus of convexity

\[ \delta_K(\varepsilon) = \min \left\{ 1 - \left\| \frac{x + y}{2} \right\|_K : x, y \in K, \|x - y\|_K \geq \varepsilon \right\} \]

- If \( \delta_K(\varepsilon) \) is big, we would say ‘highly convex’.
- The parallelogram identity
  \[ |\frac{x+y}{2}|^2 + |\frac{x-y}{2}|^2 = \frac{|x|^2 + |y|^2}{2} \]
gives the modulus of convexity for the ball,

\[ \delta_{B^n}(\varepsilon) = 1 - \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2} = \frac{1}{8} \varepsilon^2 + o(\varepsilon^2). \]
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  gives the modulus of convexity for the ball,

  \[ \delta_{B_2^n}(\varepsilon) = 1 - \sqrt{1 - \left( \frac{\varepsilon}{2} \right)^2} = \frac{1}{8} \varepsilon^2 + o(\varepsilon^2). \]

[Nordlander, 1960] For every symmetric convex body \( K \) in \( \mathbb{R}^n \),

\[ \delta_K(\varepsilon) \leq \delta_{B_2^n}(\varepsilon). \]
A symmetric convex body $K$ is called \textbf{power type $p$} if $\delta_K(\varepsilon) \gtrsim \varepsilon^p$

- $0 < p < 2$ is impossible by Nordlander.
- "convexity" increases as $p \to 2$
A symmetric convex body $K$ is called **power type** $p$ if $\delta_K(\varepsilon) \gtrsim \varepsilon^p$

- $0 < p < 2$ is impossible by Nordlander.
- "convexity" increases as $p \to 2$

[Hanner, 1956] computed the modulus of convexity for the $\ell_p$-balls.

- For $1 < p \leq 2$,
  \[
  \delta_{B^n_p}(\varepsilon) = \frac{p-1}{8} \varepsilon^2 + o(\varepsilon^2) \quad \text{(power type 2)}
  \]

- For $p \geq 2$,
  \[
  \delta_{B^n_p}(\varepsilon) = \frac{1}{2p} \varepsilon^p + o(\varepsilon^p) \quad \text{(power type } p)\]
Theorem [K.]  Let $K$ be a symmetric convex body of dimension $\geq 3$.

1. If $K$ is uniformly convex, then so is $IK$. That is,
   $$\delta_K(\varepsilon) > 0 \implies \delta_{IK}(\varepsilon) > 0.$$ 

2. If $K$ is of power type $p$, then so is $IK$. That is,
   $$\delta_K(\varepsilon) \gtrsim \varepsilon^p \implies \delta_{IK}(\varepsilon) \gtrsim \varepsilon^p.$$ 

Similar results for bodies of revolution are given in [Alfonseca-K.]

The value $p$ in the second statement is optimal.

Indeed, $IB^n_p$ contains a 2-dimensional $\ell_p$-section. So, $IB^n_p$ has the same power type of modulus of convexity as $B^n_p$. 
In fact, $IB^n_p$ contains a 2-dimensional $\ell_p$-section.

- Consider the 2-dimensional $E = \text{span}\{e_1, e_2\}$.
- Let $u \in E \cap S^{n-1}$. Take $v \in E \cap u^\perp$ (a rotation of $u$ in $E$ by $\pi/2$).
  
  $$B^n_p \cap u^\perp = B^n_p \cap (E^\perp + \text{span}\{v\}) = (B^n_p \cap E^\perp) \oplus_p (B^n_p \cap \text{span}\{v\})$$
  
  which implies $|B^n_p \cap u^\perp| = c_{n,p}|B^n_p \cap E^\perp| \cdot \rho_{B^n_p}(v)$.

- Thus, $IB^n_p \cap E = c \ B^n_p \cap E$ where $c = c_{n,p}|B^n_p \cap E^\perp|$.

\[\square\]
Uniform convexity for Intersection bodies

In fact, \( IB^p_n \) contains a 2-dimensional \( \ell_p \)-section.

- Consider the 2-dimensional \( E = \text{span} \{e_1, e_2\} \).
- Let \( u \in E \cap S^{n-1} \). Take \( v \in E \cap u^\perp \) (a rotation of \( u \) in \( E \) by \( \pi/2 \)).

\[
B^p_n \cap u^\perp = B^p_n \cap (E^\perp + \text{span} \{v\}) = (B^p_n \cap E^\perp) \oplus_p (B^p_n \cap \text{span} \{v\})
\]

which implies \( |B^p_n \cap u^\perp| = c_{n,p}|B^p_n \cap E^\perp| \cdot \rho_{B^p_n}(v) \).

- Thus, \( IB^p_n \cap E = c B^p_n \cap E \) where \( c = c_{n,p}|B^p_n \cap E^\perp| \).

\[\]

**Theorem [K.]** The double intersection body \( I^2K \) of a symmetric convex body \( K \subset \mathbb{R}^n, n \geq 3 \), is uniformly convex.

That is, every \( I^2K \) does not contain a line segment on its boundary.
Let $u_1, u_2$ be on the boundary of $IK$. We claim that $\frac{u_1 + u_2}{2} \in IK$. WLOG, assume $|u_1| = |u_2| = 1$. 

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Let $u_1, u_2$ be on the boundary of $IK$. We claim that $\frac{u_1 + u_2}{2} \in IK$. WLOG, assume $|u_1| = |u_2| = 1$.

Let $u = \frac{u_1 + u_2}{|u_1 + u_2|}$. Since

$$\left\| \frac{u_1 + u_2}{2} \right\|_{IK} = \frac{|u_1 + u_2|}{2} \left\| u \right\|_{IK} = \frac{|u_1 + u_2|}{2} / |K \cap u^\perp|,$$

we need to show $|K \cap u^\perp| \geq \frac{|u_1 + u_2|}{2}$. 
Recall: Proof of Busemann Theorem

To get $|K \cap u_1^\perp|$, integrate sections by moving with ‘uniform volume’ speed, i.e.,

$$\|1\| \cdot r_1'(t) = \text{const}, \quad \forall t \in [-1/2, 1/2].$$

Here, const is equal to 1 by the assumption $|K \cap u_1^\perp| = \|u_1\|_{IK} = 1$. 

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Here, const is equal to 1 by the assumption $|K \cap u_2^\perp| = \|u_2\|_{IK} = 1$. 
Recall: Proof of Busemann Theorem

To get the volume of $K \cap u^\perp$, integrate its sections along with the parametrization

$$r(t) = |u_1 + u_2| \frac{r_1(t)r_2(t)}{r_1(t) + r_2(t)}.$$

$\lambda = \frac{r_1}{r_1 + r_2}$

$$\|\| \geq |(1-\lambda)\| + \lambda\| \geq \|\|^{1-\lambda} \|\|^\lambda$$

$$r'(t) = |u_1 + u_2| \left[ (1-\lambda)^2 r_1' + \lambda^2 r_2' \right]$$

$$\geq |u_1 + u_2| \left[ (1-\lambda) r_1' \right]^{1-\lambda} \left[ \lambda r_2' \right]$$
Recall: Proof of Busemann Theorem

\[ |K \cap u^\perp| = \int \| \| \, dr \geq \int |(1-\lambda)\| + \lambda\| \, dr \]

\[ \geq |u_1 + u_2| \int_{-1/2}^{1/2} (\| r_1' \|^{1-\lambda} (\| r_2' \|^\lambda (1-\lambda)^{1-\lambda} \lambda^\lambda \, dt \]

\[ = |u_1 + u_2| \int_{-1/2}^{1/2} (1-\lambda)^{1-\lambda} \lambda^\lambda \, dt \]

\[ \geq \frac{1}{2} |u_1 + u_2| \]
Equality case

To hold the equality, the following should be satisfied:

1. Equality in the AM/GM inequality should hold:
   \[ \lambda(t) = \frac{1}{2}, \text{ so } r_1(t) = r_2(t) \text{ for all } t \in [-1/2, 1/2]. \]

2. Equality in the Brun-Minkowski inequality should hold:
   the sections \( I \) and \( J \) are homothetic each other.

3. \( I = \frac{1}{2}(I + J) \), the equality case of \( I \supset (1 - \lambda)I + \lambda J \).

In this case three sections \( I, J, K \) are congruent.
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2. Equality in the Brun-Minkowski inequality should hold:
   the sections are homothetic each other.

3. \[ \underline{l} = \frac{1}{2}(\underline{l} + \underline{l}) \], the equality case of \( \underline{l} \supset (1 - \lambda)\underline{l} + \lambda\underline{l} \).

In this case three sections \( \underline{l}, \underline{l}, \underline{l} \) are congruent.

\[ u_1 \quad \underline{IK} \quad u_2 \]

implies

\[ IK \quad K \]
Let $K$ be a symmetric convex body in $\mathbb{R}^n$, $n \geq 3$. Then the double intersection body $I^2K$ of $K$ does not contain a line segment on its boundary, (i.e., $I^2K$ is uniformly convex).

Suppose that $I^2K$ contains a line segment on its boundary.
Let $K$ be a symmetric convex body in $\mathbb{R}^n$, $n \geq 3$. Then the double intersection body $I^2 K$ of $K$ does not contain a line segment on its boundary, (i.e., $I^2 K$ is uniformly convex).

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Contradiction!
Question 1: Quantify the previous statement.

That is, what happens on $K$ if $IK$ has ‘almost flat’ chord on its boundary?
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That is, what happens on $K$ if $IK$ has ‘almost flat’ chord on its boundary?

Question 2: Prove or disprove that

$$\delta_{I^2K}(\varepsilon) \geq \delta_K(\varepsilon) \quad \forall \varepsilon > 0$$

and the equality holds if and only if $K$ is an ellipsoid.
THANK YOU!