# ‘Convexity’ of Intersection Bodies

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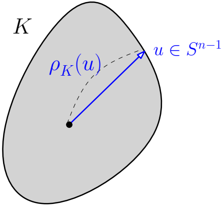
*AIM Workshop on Sections of Convex Bodies,*
*Palo Alto, August 5-9, 2013*

# Basic notions : Radial function

- The radial function $\rho_K$ of a body $K$ is defined by



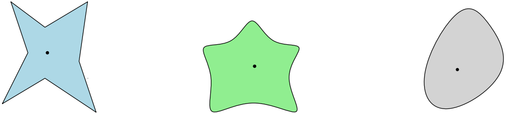

$$\rho_K(u) = \max\{\lambda \geq 0 : \lambda u \in K\}, \quad u \in S^{n-1}.$$

Note also that

$$\rho_K(u) = \|u\|_K^{-1} \quad \text{for } u \in S^{n-1},$$

where $\|\cdot\|_K$ is the Minkowski functional of $K$.

- A star body is a body whose radial function is positive and continuous.



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In 1988, **Lutwak** introduced the notion of intersection bodies.

Let $K$ be a star body in $\mathbb{R}^n$. The intersection body of $K$, denoted by $IK$, is a body whose radial function is

$$\rho_{IK}(u) = |K \cap u^\perp| \quad \text{for each } u \in S^{n-1}.$$

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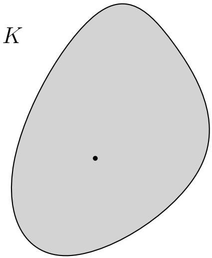


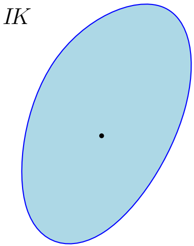

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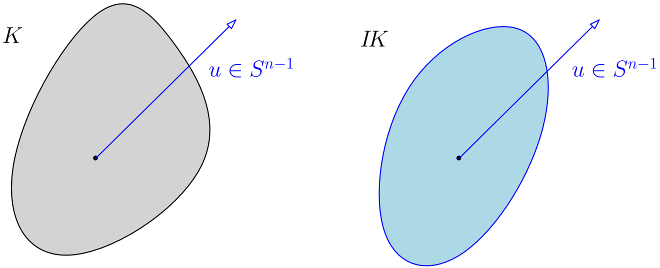

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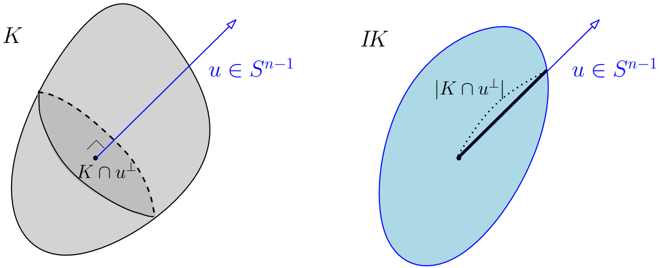
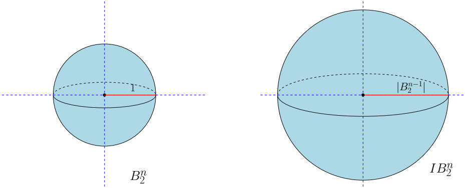


# Intersection bodies: Examples

- For the ball $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$, $IB_2^n = cB_2^n$ for $c = |B_2^{n-1}|$.



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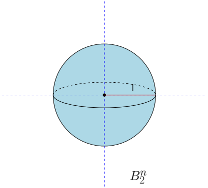
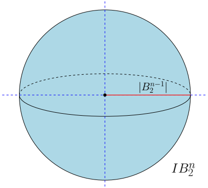
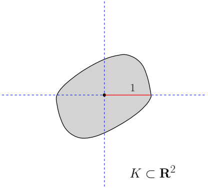
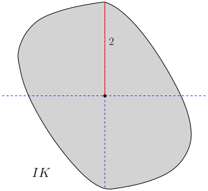



- For $K \subset \mathbb{R}^2$ symmetric, $IK$ is a rotation of $2K$ by angle $\pi/2$.



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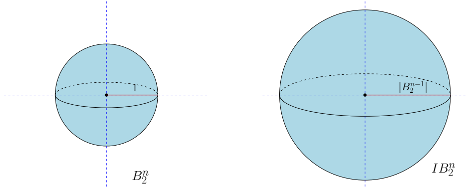


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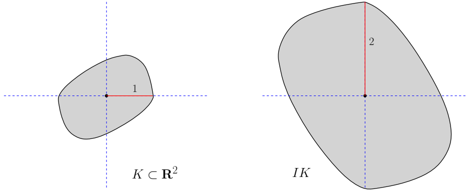


- For $T \in \mathrm{GL}(n)$,

$$I(TK) = |\det T| \, (T^{-1})^*(IK).$$

# Intersection bodies: motivations

- Related to the solution for the **Busemann-Petty problem**: For symmetric convex bodies $K$, $L$ in $\mathbb{R}^n$, is it true that

$$|K \cap H| \geq |L \cap H| \quad \text{for all hyperplanes } H \quad \Rightarrow \quad |K| \geq |L|?$$

[ yes if $n \leq 4$, no if $n \geq 5$ ]. Larman, Rogers, Ball, Lutwak, Gardner, Zhang, Koldobsky, Schlumprecht.

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- Connection with the **Spherical Radon transform** $\mathcal{R}$.

$$\begin{aligned} \rho_{IK}(\theta) &= |K \cap \theta^\perp| = \int_{S^{n-1}\cap\theta^\perp} \int_0^{\rho_K(u)} r^{n-2}\, dr du \\ &= \frac{1}{n-1} \int_{S^{n-1}\cap\theta^\perp} \rho_K^{n-1}(u) du = \frac{1}{n-1} \mathcal{R}\rho_K^{n-1}(\theta), \end{aligned}$$

  that is, $\rho_{IK} = \frac{1}{n-1} \mathcal{R}\rho_K^{n-1}$.

# Busemann's Theorem

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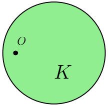

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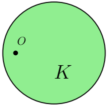


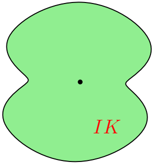

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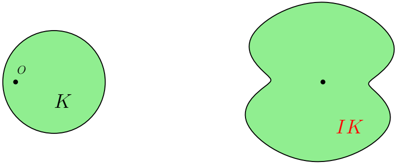


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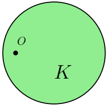

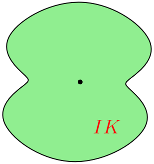


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**Main Question**: **How much** of 'convexity' is preserved under the intersection body operator $I$?

# How to measure 'convexity'

(1) **Quasi-convexity**

$K$ is $q$-convex if $t^{\frac{1}{q}}x + (1-t)^{\frac{1}{q}}y \in K$ whenever $x, y \in K$, $t \in [0,1]$
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(2) **Banach-Mazur distance from $B_2^n$**

$$d_{BM}(K, B_2^n) = \min\left\{r : B_2^n \subset TK \subset rB_2^n, T \in GL(n)\right\}$$

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"convexity" increases as $d_{BM}(K, B_2^n) \to 1$.

(3) **Modulus of convexity**

$$\delta_K(\varepsilon) = \min\left\{1 - \frac{1}{2}\|x+y\|_K : x, y \in K, \|x-y\|_K \geq \varepsilon\right\}$$

"convexity" increases as $\delta_K(\varepsilon)$ does.

# (1) Quasi-convexity

Let $0 < q \leq 1$. A star body $K \subset \mathbb{R}^n$ is called *q-convex* if

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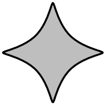

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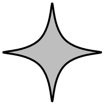

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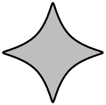

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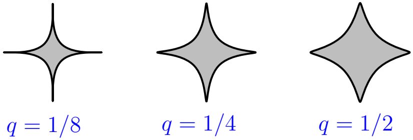

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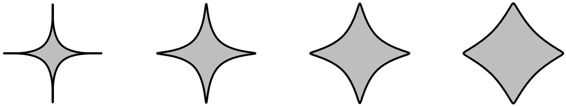

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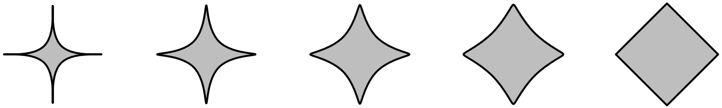

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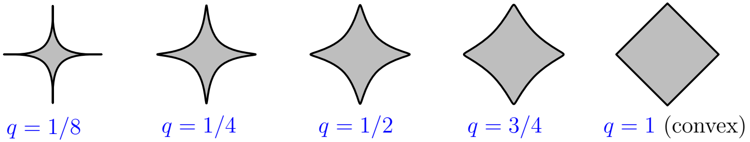


**Question** If $K$ is $q$-convex, for which $q'$ the intersection body $IK$ is $q'$-convex?

# (1) Quasi-convexity for Intersection bodies

**Theorem** [K.-Yaskin-Zvavitch, 2011]
Let $K$ be a symmetric star body in $\mathbb{R}^n$ and $0 < q \leq 1$. Then, if $K$ is $q$-convex, $IK$ is $q'$-convex where

$$q' = \left[(1/q - 1)(n-1) + 1\right]^{-1}$$

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**More generally**, for a log-concave measure $\mu$ on $\mathbb{R}^n$,

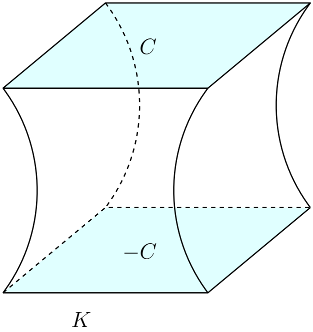
Consider the intersection body $I_\mu K$ w.r.t. $\mu$ defined by

$$\rho_{IK}(u) = \mu(K \cap u^\perp) \quad \forall u \in S^{n-1}.$$

Then the above theorem still holds for $I_\mu K$.

# Quai-convexity: Example

Let $K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y : x \in C, y \in -C, 0 \le t \le 1 \right\}$,
where $C = \{(1, x_2, \ldots, x_n) : |x_2|, \ldots, |x_n| \le 1\}$.

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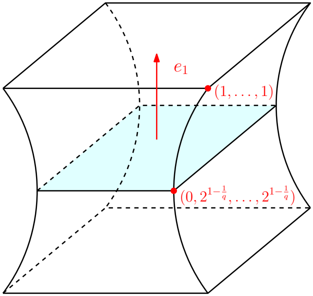


$|K \cap e_1^{\perp}| = 2^{(2-1/q)(n-1)}$

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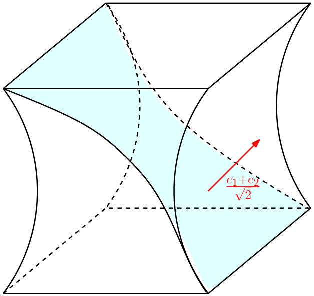


$|K \cap (\frac{e_1 \pm e_2}{\sqrt{2}})^\perp| \succeq 2^{n-1/2}$

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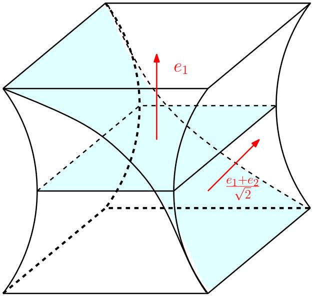


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- $K$ is $q$-convex.
- $\|e_1\|_{IK} = 2^{(1/q-2)(n-1)}$,
  $\|e_1 \pm e_2\|_{IK} \lesssim 2^{1-n}$.

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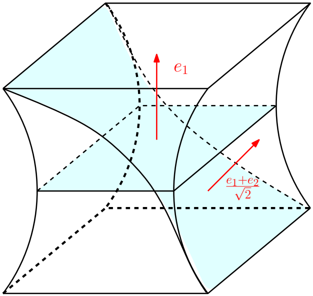


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- $K$ is $q$-convex.
- $\|e_1\|_{IK} = 2^{(1/q-2)(n-1)}$,
  $\|e_1 \pm e_2\|_{IK} \lesssim 2^{1-n}$.
- If $IK$ is $q'$-convex, then

  $$\|2e_1\|_{IK}^{q'} \le \|e_1 + e_2\|_{IK}^{q'} + \|e_1 - e_2\|_{IK}^{q'}$$

  gives

  $$q' \lesssim [(1/q-1)(n-1)+1]^{-1}.$$

# (2) Banach-Mazur distance from $B_2^n$

[**Hensley**, 1980] For any symmetric convex body $K$ in $\mathbb{R}^n$,

$$d_{BM}(IK, B_2^n) \leq C$$

where $C > 1$ is a universal constant.

- Note that $d_{BM}(B_\infty^n, B_2^n) = \sqrt{n}$. In fact, a lot of symmetric convex bodies are very far from the ball in high dimension.
- Nevertheless, the above theorem says that their intersection bodies should be bounded from the ball by an absolute constant.

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[**Hensley**, 1980] For any symmetric convex body $K$ in $\mathbb{R}^n$,

$$d_{BM}(IK, B_2^n) \leq C$$

where $C > 1$ is a universal constant.

- Note that $d_{BM}(B_\infty^n, B_2^n) = \sqrt{n}$. In fact, a lot of symmetric convex bodies are very far from the ball in high dimension.
- Nevertheless, the above theorem says that their intersection bodies should be bounded from the ball by an absolute constant.

**Questions** [**Lutwak**, 1990's] Are the followings true?

- $d_{BM}(IK, B_2^n) \leq d_{BM}(K, B_2^n)$,
- $d_{BM}(IK, K) = 1 \Rightarrow d_{BM}(K, B_2^n) = 1$,
- $d_{BM}(I^m K, B_2^n) \to 1$ as $m \to \infty$.

# Partial answers

[**Fish-Nazarov-Ryabogin-Zvavitch**, 2011] If a star body $K$ is close enough to the ball $B_2^n$, then

$$d_{BM}(I^m K, B_2^n) \to 1 \quad \text{as } m \to \infty.$$

# Partial answers

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**Theorem** [**Alfonseca-K.**, 2013]
For every $\varepsilon > 0$, there exists an integer $N \geq 1$ such that for every $n \geq N$ and any convex body $K$ of revolution in $\mathbb{R}^n$,

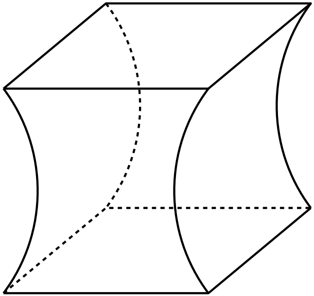
$$d_{BM}(I^2 K, B_2^n) \leq 1 + \varepsilon.$$

- Roughly speaking, the double intersection body of a body of revolution is close enough to an ellipsoid in high dimension.
- But the single intersection body is not enough for the above theorem.

# Negative answer for star bodies

$d_{BM}(IK, B_2^n) \leq d_{BM}(K, B_2^n)$ may not be true without convexity

Consider $K = \left\{ t^{\frac{1}{q}} x + (1-t)^{\frac{1}{q}} y : x \in C, y \in -C, 0 \leq t \leq 1 \right\}$,



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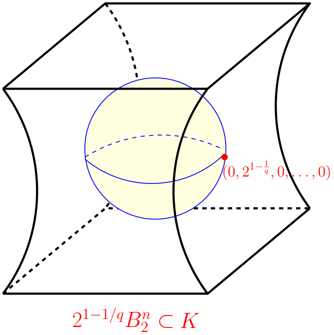

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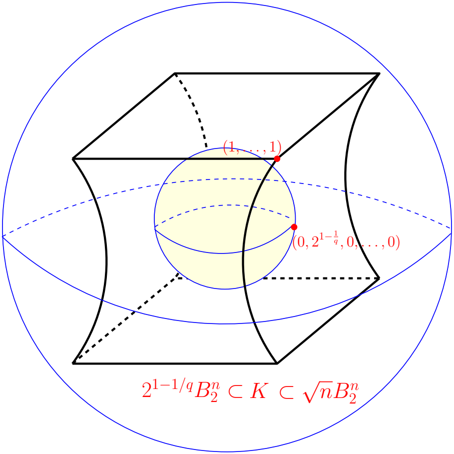


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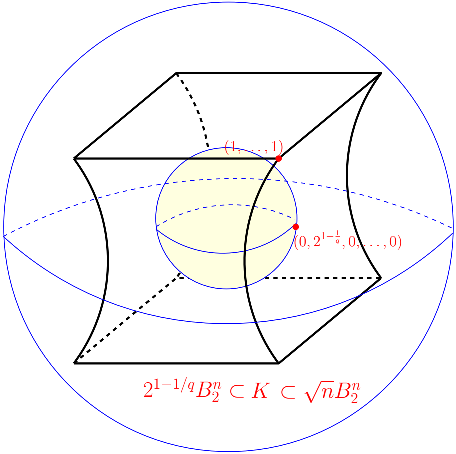
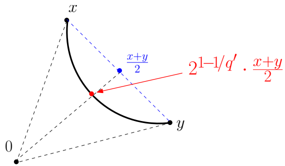



- $d_{BM}(K, B_2^n) \leq 2^{1/q-1}\sqrt{n}$.
- Note that $IK$ is $q'$-convex where $q' = [(1/q-1)(n-1)+1]^{-1}$.

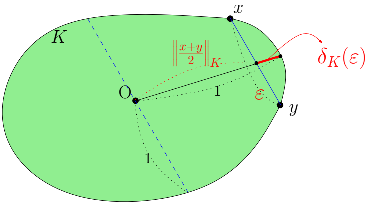



So, $d_{BM}(IK, B_2^n) \geq 2^{(1/q-1)(n-1)}$.

- $d_{BM}(IK, B_2^n) \gg d_{BM}(K, B_2^n)$.

# (3) Modulus of convexity

$$\delta_K(\varepsilon) = \min\left\{1 - \left\|\frac{x+y}{2}\right\|_K : x, y \in K, \|x - y\|_K \geq \varepsilon\right\}$$




- If $\delta_K(\varepsilon)$ is big, we would say 'highly convex'.
- The parallelogram identity $\left|\frac{x+y}{2}\right|^2 + \left|\frac{x-y}{2}\right|^2 = \frac{|x|^2+|y|^2}{2}$ gives the modulus of convexity for the ball,

$$\delta_{B_2^n}(\varepsilon) = 1 - \sqrt{1 - \left(\frac{\varepsilon}{2}\right)^2} = \frac{1}{8}\varepsilon^2 + o(\varepsilon^2).$$

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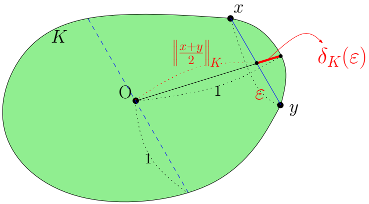


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[**Nordlander**, 1960] For every symmetric convex body $K$ in $\mathbb{R}^n$,

$$\delta_K(\varepsilon) \leq \delta_{B_2^n}(\varepsilon).$$

# Uniform convexity: power type

A symmetric convex body $K$ is called power type $p$ if $\delta_K(\varepsilon) \gtrsim \varepsilon^p$

- $0 < p < 2$ is impossible by Nordlander.
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[**Hanner**, 1956] computed the modulus of convexity for the $\ell_p$-balls.

- For $1 < p \leq 2$,

$$\delta_{B_p^n}(\varepsilon) = \frac{p-1}{8}\,\varepsilon^2 + o(\varepsilon^2) \quad (\text{power type } 2)$$

- For $p \geq 2$,

$$\delta_{B_p^n}(\varepsilon) = \frac{1}{2p}\,\varepsilon^p + o(\varepsilon^p) \quad (\text{power type } p)$$

# Uniform convexity for Intersection bodies

**Theorem** [K.] Let $K$ be a symmetric convex body of dimension $\geq 3$.

1. If $K$ is uniformly convex, then so is $IK$. That is,

$$\delta_K(\varepsilon) > 0 \quad \Rightarrow \quad \delta_{IK}(\varepsilon) > 0.$$

2. If $K$ is of power type $p$, then so is $IK$. That is,

$$\delta_K(\varepsilon) \gtrsim \varepsilon^p \quad \Rightarrow \quad \delta_{IK}(\varepsilon) \gtrsim \varepsilon^p.$$

- Similar results for bodies of revolution are given in [Alfonseca-K.]
- The value $p$ in the second statement is optimal.

  Indeed, $IB_p^n$ contains a 2-dimensional $\ell_p$-section. So, $IB_p^n$ has the same power type of modulus of convexity as $B_p^n$.

# Uniform convexity for Intersection bodies

In fact, $IB_p^n$ contains a 2-dimensional $\ell_p$-section.

- Consider the 2-dimensional $E = \mathrm{span}\,\{e_1, e_2\}$.
- Let $u \in E \cap S^{n-1}$. Take $v \in E \cap u^\perp$ (a rotation of $u$ in $E$ by $\pi/2$).

$$B_p^n \cap u^\perp = B_p^n \cap (E^\perp + \mathrm{span}\,\{v\}) = (B_p^n \cap E^\perp) \oplus_p (B_p^n \cap \mathrm{span}\,\{v\})$$

  which implies $|B_p^n \cap u^\perp| = c_{n,p}|B_p^n \cap E^\perp| \cdot \rho_{B_p^n}(v)$.
- Thus, $IB_p^n \cap E = c\, B_p^n \cap E$ where $c = c_{n,p}|B_p^n \cap E^\perp|$. □

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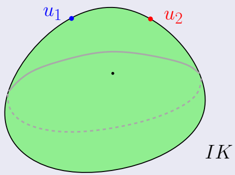
**Theorem [K.]** The double intersection body $I^2K$ of a symmetric convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is uniformly convex.

That is, every $I^2K$ does not contain a line segment on its boundary.

# Recall: Proof of Busemann Theorem




1. Let $u_1$, $u_2$ be on the boundary of $IK$. We claim that $\frac{u_1+u_2}{2} \in IK$. WLOG, assume $|u_1| = |u_2| = 1$.

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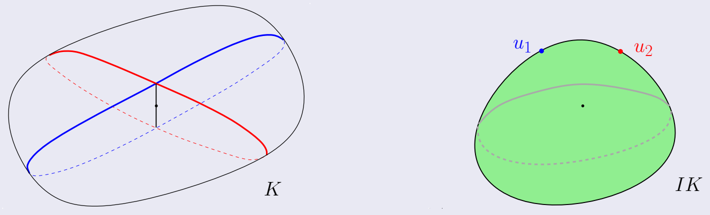


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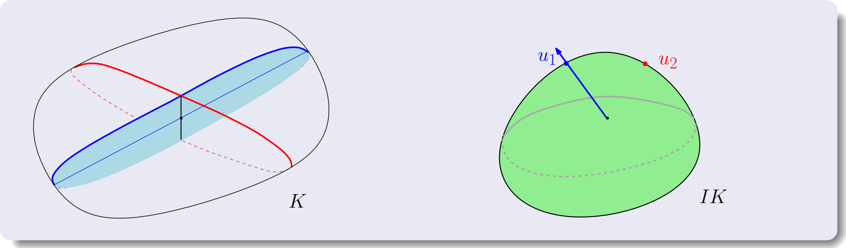


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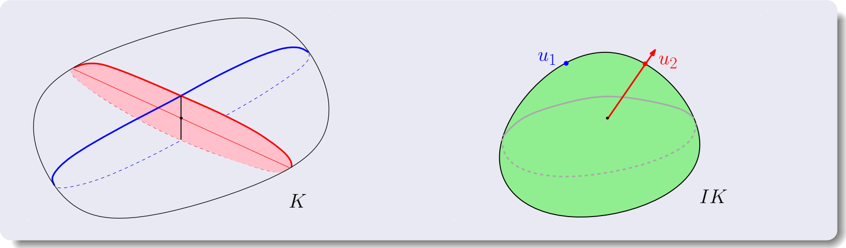


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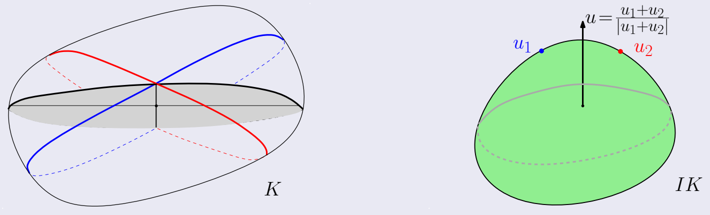


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2. Let $u = \frac{u_1+u_2}{|u_1+u_2|}$. Since

$$\left\|\frac{u_1+u_2}{2}\right\|_{IK} = \frac{|u_1+u_2|}{2}\|u\|_{IK} = \frac{|u_1+u_2|}{2}/|K \cap u^{\perp}|,$$

we need to show $|K \cap u^{\perp}| \geq \frac{|u_1+u_2|}{2}$.

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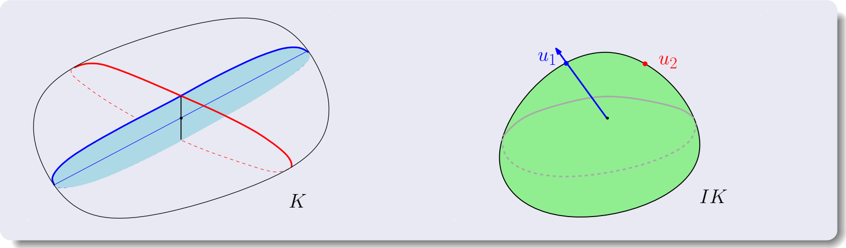


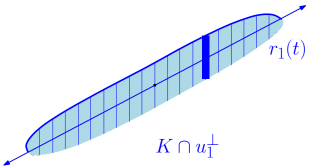


To get $|K \cap u_1^\perp|$, integrate sections by moving with **'uniform volume' speed**, i.e.,

$$\left|\,\blacksquare\,\right| \cdot r_1'(t) = \mathrm{const}, \ \forall t \in [-1/2, 1/2].$$

Here, const is equal to 1 by the assumption $|K \cap u_1^\perp| = \|u_1\|_{IK} = 1$.

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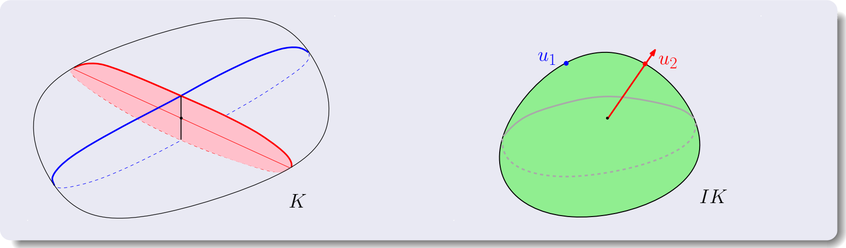


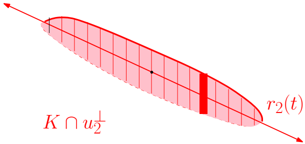


To get $|K \cap u_2^{\perp}|$, integrate sections by moving with **'uniform volume' speed**, i.e.,

$$\big|\,\blacksquare\,\big| \cdot r_2'(t) = \mathrm{const}, \ \forall t \in [-1/2, 1/2].$$

Here, $\mathrm{const}$ is equal to 1 by the assumption $|K \cap u_2^{\perp}| = \|u_2\|_{IK} = 1$.

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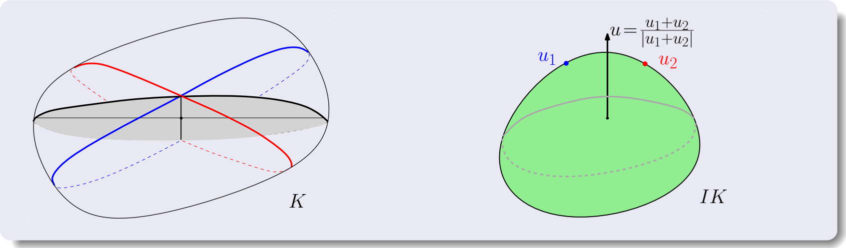


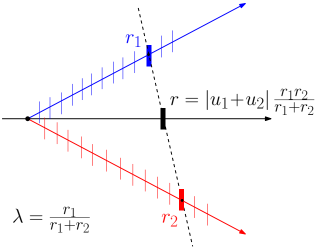


To get the volume of $K \cap u^{\perp}$, integrate its sections along with the parametrization

$$r(t) = |u_1+u_2| \frac{r_1(t) r_2(t)}{r_1(t) + r_2(t)}.$$

- $|▮| \geq |(1-\lambda)▮ + \lambda▮| \geq |▮|^{1-\lambda} \, |▮|^{\lambda}$
- $r'(t) = |u_1+u_2| \left[(1-\lambda)^2 r_1' + \lambda^2 r_2'\right] \geq |u_1+u_2| \left[(1-\lambda) r_1'\right]^{1-\lambda} \left[\lambda r_2'\right]^{\lambda}$

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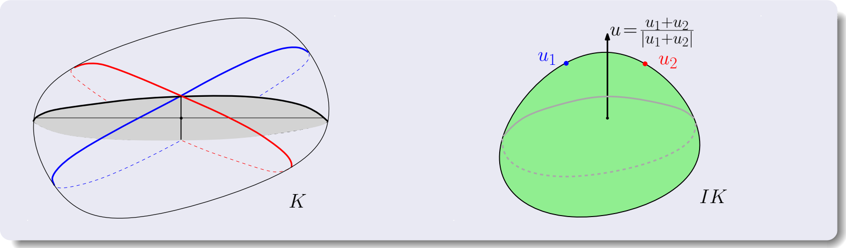


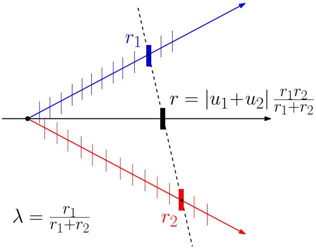


$$|K \cap u^{\perp}| = \int |\blacksquare|\, dr \geq \int |(1-\lambda)\blacksquare + \lambda \blacksquare|\, dr$$

$$\geq |u_1+u_2| \int_{-1/2}^{1/2} (|\blacksquare|\, r_1')^{1-\lambda} (|\blacksquare|\, r_2')^{\lambda} (1-\lambda)^{1-\lambda} \lambda^{\lambda} dt$$

$$= |u_1+u_2| \int_{-1/2}^{1/2} (1-\lambda)^{1-\lambda} \lambda^{\lambda} dt$$

$$\geq \frac{1}{2} |u_1+u_2|$$

# Equality case

To hold the equality, the following should be satisfied:

1. Equality in the AM/GM inequality should hold:
   $\boxed{\lambda(t) = \frac{1}{2}}$, so $r_1(t) = r_2(t)$ for all $t \in [-1/2, 1/2]$.
2. Equality in the Brun-Minkowski inequality should hold:
   the sections $\boxed{{\color{blue}\rule{2pt}{8pt}} \text{ and } {\color{red}\rule{2pt}{8pt}} \text{ are homothetic each other}}$.
3. $\boxed{\rule{2pt}{8pt} = \frac{1}{2}({\color{blue}\rule{2pt}{8pt}} + {\color{red}\rule{2pt}{8pt}})}$, the equality case of $\rule{2pt}{8pt} \supset (1-\lambda){\color{blue}\rule{2pt}{8pt}} + \lambda{\color{red}\rule{2pt}{8pt}}$.

In this case three sections $\rule{2pt}{8pt}$, ${\color{blue}\rule{2pt}{8pt}}$, ${\color{red}\rule{2pt}{8pt}}$ are congruent.

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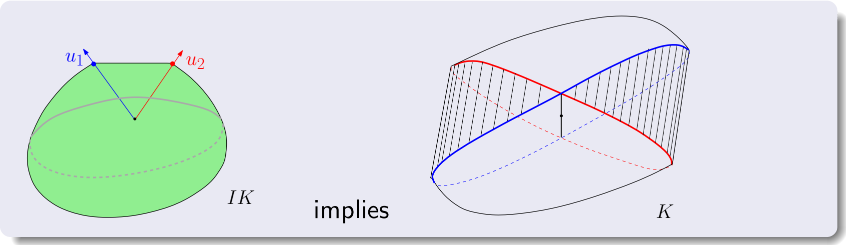
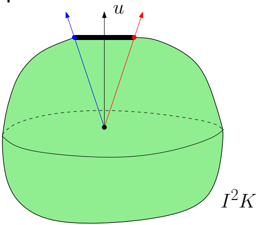


# Uniform convexity for $I^2K$

Let $K$ be a symmetric convex body in $\mathbb{R}^n$, $n \geq 3$. Then the double intersection body $I^2K$ of $K$ does not contain a line segment on its boundary, (i.e., $I^2K$ is uniformly convex).

Suppose that $I^2K$ contains a line segment on its boundary.



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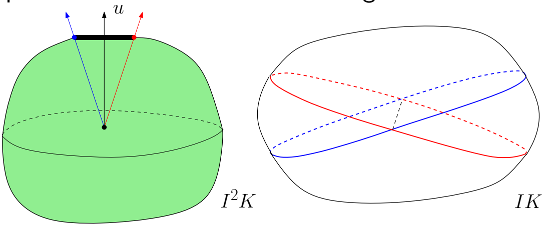

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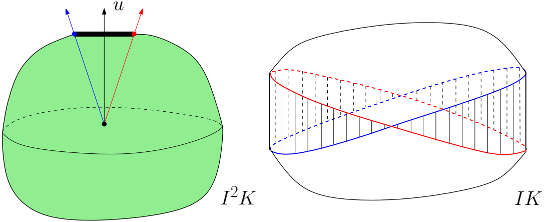

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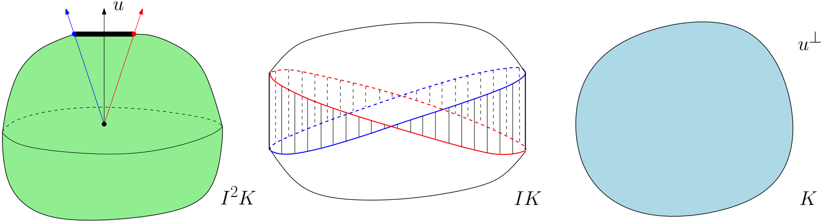

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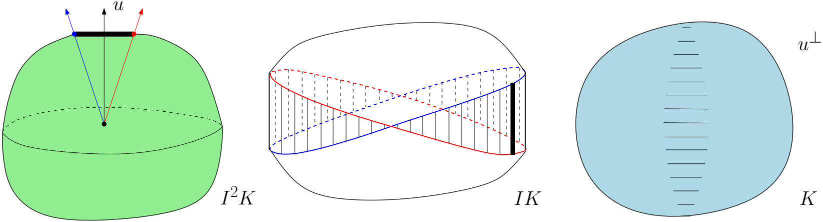

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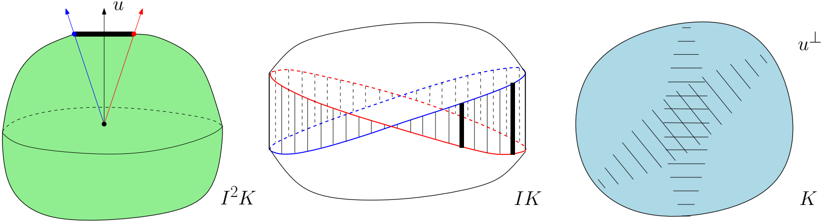

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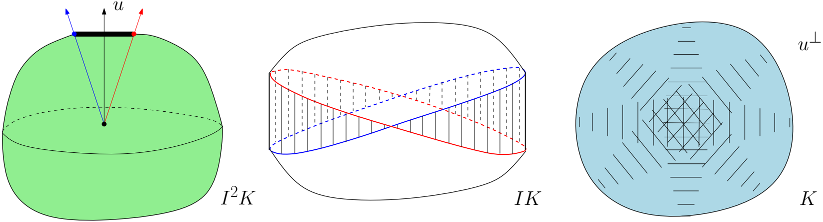

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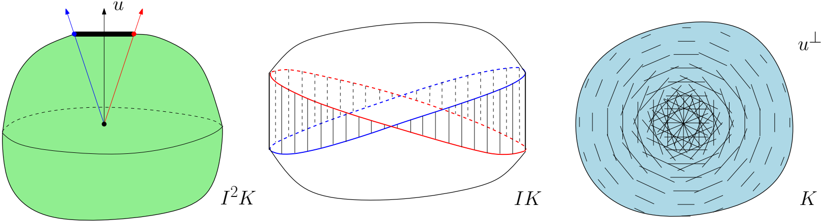


Contradiction! □

# Open questions

**Question 1**: Quantify the previous statement.

That is, what happens on $K$ if $IK$ has 'almost flat' chord on its boundary?

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**Question 2**: Prove or disprove that

$$\delta_{I^2K}(\varepsilon) \geq \delta_K(\varepsilon) \quad \forall \varepsilon > 0$$

and the equality holds if and only if $K$ is an ellipsoid.

THANK YOU!