Vertex index of symmetric convex bodies

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based on joint works with
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Motivation

Covering Conjecture (Gohberg-Markus, 60)

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In other words, for every convex body $K \subset \mathbb{R}^d$ there exists $0 < \lambda < 1$ and points $x_i \in \mathbb{R}^d$, $i \leq 2^d$, such that

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Remark 2. The best known bound is $7d(\ln d)\, 2^d$ in the symmetric case and $4\sqrt{d}(\ln d)4^d$ in the general case.

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Let K be a convex body in $\mathbb{R}^d$ with non-empty interior.

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Def. 2. A family of exterior points of K, $\{p_1, p_2, \ldots, p_m\} \subset \mathbb{R}^d \setminus K$, illuminates K if each boundary point of K is illuminated by at least one of p_i's.

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Remark 2. Two conjectures above are equivalent.

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This insures that far-away points of illumination are penalized.

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K. Bezdek posed the problem of finding the upper bound for the $\mathrm{ill}(K)$. He also provided some estimates and conjectured that for every 0-symmetric body K

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Motivated by the notion of the illumination parameter K. Swanepoel (2004) introduced the *covering parameter* of a convex body K by

$$\text{cov}(K) = \inf \left\{ \sum_i \frac{1}{1-\lambda_i} \;\middle|\; K \subset \bigcup_i (x_i + \lambda_i K), 0 < \lambda_i < 1, x_i \in \mathbb{R}^d \right\}.$$

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Theorem (Swanepoel)

For every symmetric convex body K in $\mathbb{R}^d$ one has

$$\mathrm{ill}(K) \leq 2\,\mathrm{cov}(K) \leq C\,2^d d^2 \ln d.$$

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Simple properties

Below K, L are symmetric convex bodies, T is an invertible linear operator, $d(\cdot,\cdot)$ denotes the Banach-Mazur distance, that is

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Remark. Note that $\text{ill}(B_\infty^d) = 2^d$, while below we will see that $\text{vein}(K) \le Cd^{3/2}$. It shows that $\text{ill}(K)$ is rather unstable, while Claim 1 shows that $\text{vein}(K)$ is stable.

Results

Theorem

For every symmetric convex body K in $\mathbb{R}^d$ one has

$$\frac{d^{3/2}}{\sqrt{2\pi e}\,\mathrm{ovr}(K)} \le \mathrm{vein}(K) \le 24\, d^{3/2}.$$

Here $\mathrm{ovr}(K)$ is the outer volume ratio of K, $\mathrm{ovr}(K) = \inf\left(\mathrm{Vol}(\mathcal{E})/\,\mathrm{Vol}(K)\right)^{1/d}$, where the infimum is taken over all ellipsoids $\mathcal{E} \supset K$ and $\mathrm{Vol}(\cdot)$ denotes the volume.

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3. The lower bound is sharp for every ℓ_p unit ball. Note, for $p \ge 2$: $\mathrm{ovr}(B_p^d) \le C$ (bounded outer volume ratio) and for $1 \le p \le 2$: $\mathrm{ovr}(B_p^d) \approx d^{1/p-1/2}$.

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Thus

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3. We do not know the best possible upper estimate in the 3-dimensional case.

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Assume $m < 2d$. Since $K = -K \subset P$, we have $\|x\|_K \geq \|-x\|_P$ for every $x \in \mathbb{R}^d$. Therefore, applying our Theorem, we obtain

$$\sum_{i=1}^{m} \|p_i\|_K \geq \sum_{i=1}^{m} \|-p_i\|_P \geq \frac{m^2}{2k} = \frac{(d+k)^2}{2k} \geq 2d.$$

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Using Claim 1 (or just direct computations) we obtain

$$\mathrm{vein}(K) \le 3\sqrt{d}\,\mathrm{vein}(P) \le 24d^{3/2}.$$

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Finally, since B_2^d is the minimal volume ellipsoid for K and $\|\cdot\|_K \ge |\cdot|$, we have

$$\frac{1}{\operatorname{ovr}(K)} = \left(\frac{\operatorname{vol}(K)}{\operatorname{vol}\left(B_2^d\right)} \right)^{1/d} \le \left(\operatorname{vol}\left(B_2^d\right)\right)^{1/d} \frac{1}{d} \sum_{i=1}^N \|p_i\|_K \le \frac{\sqrt{2\pi e}}{d^{3/2}} \sum_{i=1}^N \|p_i\|_K.$$

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Theorem. If $K = \text{conv}\{x_i\}_{i \le m} \subset \mathbb{R}^d$ with $m = k + d \le 2d$ then

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Therefore

$$\|z\|_{S^{\circ}} := \begin{cases} \sum_{j=1}^{m} \langle z, e_j \rangle & \text{if } \langle z, e_j \rangle \ge 0 \text{ for every } j \le m, \\ \infty & \text{otherwise.} \end{cases}$$

Proof

Then $\|z\|_{PS^\circ} = \inf_{y\in L} \|z+y\|_{S^\circ}$

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For such matrices (**Lemma**) one has

$$A = \sum_{i=1}^m \sum_{j=1}^m y_{ij} - m \geq \frac{m(m-1)}{2k-1} \geq \frac{m^2}{2k}.$$

Proof of Lemma.

Lemma. Let $\Lambda = \{\lambda_{ij}\}$ be an $m \times m$ matrix of rank k with nonnegative entries such that $\lambda_{ii} \geq 1$ for every $i \leq m$. Then $\sum_{i,j} \lambda_{ij} \geq 3m - 2k$ and if $m \geq 2k$ then

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Thus, using Weil's Theorem,

$$\sum_{i,j=1}^{m} t_{ij} \geq \sum_{i=1}^{m} |\lambda_i| \geq 2m - 2k.$$

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Using averaging argument, we obtain

$$\sum_{i,j=1}^{m} \lambda_{ij} = m + \sum_{\substack{i,j=1 \\ i \neq j}}^{m} \lambda_{ij} = m + \binom{m-2}{l-2}^{-1} \sum_{\substack{\sigma \subset \{1,2,\dots,m\} \\ |\sigma| = \ell}} \sum_{\substack{i,j \in \sigma \\ i \neq j}} \lambda_{ij}$$

$$\geq m + \binom{m-2}{l-2}^{-1} \binom{m}{l} (2l - 2k) = m + 2\frac{m(m-1)}{\ell(\ell-1)} (\ell - k) .$$

Proof

Thus, since $T = \Lambda - I$, we observe $\sum_{i,j} \lambda_{ij} \geq 3m - 2k$.

Now assume $m \geq 2k$. Let $2k \leq \ell \leq m$ and $\sigma \subset \{1, 2, \dots, m\}$ be of cardinality ℓ, and

$$\bar{\Lambda} = \{\lambda_{ij}\}_{i,j \in \sigma} .$$

Clearly, rank $\bar{\Lambda} \leq k$, so, by the first part,

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The choice $\ell = 2k$ completes the proof.

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Almost nothing is known in the non-symmetric case, even how large N should be taken in order to get $\lambda \le d^{1-\varepsilon}$. We conjecture that $N = Cd$ is enough for $\lambda \le C\sqrt{d}$ (as in the symmetric case).

Lower bound for the Euclidean ball

The above proof gives $\text{vein}(B_2^d) \geq d^{3/2}/\sqrt{2\pi e}$. Here we suggest another approach to the problem, which is of independent interest, and leads to the bound $d^{3/2}/\sqrt{3}$.

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(recall $(X \otimes Y)(z) = \langle X, z \rangle Y$, or $X \otimes Y = \{Y_i X_j\}$). Note that the rank of S equals d.

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Let $s_1 \geq s_2 \geq \ldots \geq s_d > 0$ be the singular values of S and let $\{w_i\}_{i \leq n}$, $\{z_i\}_{i \leq d}$ be orthonormal systems such that $S = \sum_{i=1}^{d} s_i w_i \otimes z_i$.

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Considering the extreme points of the section of the cube $B_\infty^N \cap E_m$ we observe that there exists a vector $y = \{y_i\}_{i \leq N} \in B_\infty^N \cap E_m$ such that the set $A = \{i \mid |y_i| = 1\}$ has cardinality at least $d + 1 - m$.

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WLOG we assume that $|A| = d+1-m$ (otherwise pass to a subset of A). Then

$$|(a\Lambda)^{-1}y| = \frac{1}{a}\sqrt{\sum_{i=1}^{N} \frac{y_i^2}{\lambda_i^2}} \geq \frac{1}{a}\sqrt{\sum_{i \in A} \frac{1}{\lambda_i^2}} \geq \frac{d+1-m}{a\sqrt{\sum_{i \in A} \lambda_i^2}} \geq \frac{d+1-m}{a\sqrt{\sum_{i=1}^N \lambda_i^2}} = \frac{d+1-m}{a}.$$

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Note that by construction $y \in E_m \subset \operatorname{Im} T^*$, so denoting the inverse of T^* from the image by $(T^*)^{-1}$ we have

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This shows $s_m \geq (d+1-m)/a$ and implies

$$\frac{d^3}{3a^2} \leq \frac{1}{a^2} \sum_{m=1}^{d} (d+1-m)^2 \leq \sum_{m=1}^{d} s_m^2 = 1,$$

which proves the desired result.