Combinatorial challenges in toric varieties

organized by
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Workshop Summary

A lattice polytope $P \subset \mathbb{R}^d$ is the convex hull of finitely many points in the lattice $\mathbb{Z}^d$. The polytope P is smooth if the corresponding projective toric variety is smooth. Combinatorially, P is smooth if it simple, and for each vertex the first lattice vectors along each edge are part of a lattice basis of $\mathbb{Z}^d$. This workshop concentrated on the following questions:

Question 1: For which lattice polytopes P is every lattice point in the dilation kP a sum of k lattice points in P? Is this guaranteed by insisting that the polytope be smooth? This property means that the corresponding projective toric variety is projectively normal.

Question 2: Let $S = \Bbbk[x_u : u \in P \cap \mathbb{Z}^d]$. Let $I_P \subseteq S$ be the toric ideal defining the projective embedding of X_P given by P, so $R_P := S/I_P$ is the semigroup algebra for the monoid generated by the vectors $\{(u, 1) \in \mathbb{Z}^{d+1} : u \in \mathbb{Z}^d \cap P\}$.

Is I_P generated in degree 2?

Question 3: Is $R_P = S/I_P$ a Koszul algebra? This means that the residue field of R_P has a linear free resolution as an R_P-module. Koszulness of R_P is implied by the existence of a quadratic Gröbner basis for I_P. In turn, Koszulness implies quadratic generation of I_P.

Question 4: For any P there is a constant c for which the dilation cP has a unimodular triangulation. Is there a constant $c(d)$ so that for any polytope P of dimension d the dilation cP has a unimodular triangulation for any $c \geq c(d)$?

Workshop activities. Monday morning began with an introduction to these problems by the organizers. The afternoon started with a large group session of approaches to the problems, followed by breaking into working groups. On Tuesday morning we heard from Milena Hering on guaranteeing the Koszul property by taking suitable multiples of P, and from Winfried Bruns on his automated search for counterexamples to these conjectures. Wednesday morning Hidefumi Ohsugi spoke about a construction of a polytope from the edges of a (hyper-)graph, and discussed curious examples obtained by this method. This was followed by Sam Payne, who explained the combinatorial interpretation of Frobenius splitting as applied to these problems. On Thursday morning Alan Stapledon discussed how Kneser's theorem from additive number theory yields new inequalities for the Ehrhart h^*-vector. Finally, on Friday morning we heard from Bill Fulton on Chow groups of toric varieties and their equivariant versions, and Francisco Santos on unimodular triangulations of dilated polytopes in $\mathbb{R}^3$.

After the morning talks each day we had reports from the previous day's working groups (either before or after lunch) then broke into smaller working groups that discussed approaches to the problems or summarized known results.

A group which named itself the "Zoo" met several days to discuss classes of polytopes for which affirmative answers were known to these problems. A related group, which also met several days, discussed combinatorially motivated classes, such as matroid polytopes and Gelfand-Tsetlin polytopes, and also relations to the Integer Caratheodory Property (ICP). Some of this group continued to discuss proof techniques that had arisen during the week, such as the notion of strong connectivity, and the class of "locally totally unimodular" polytopes (not-necessarily simple polytopes whose edge directions at each vertex form a totally unimodular matrix). Another group discussed practical aspects in the search for counterexamples. Winfried Bruns explained his code which generates smooth polytopes and tests them for normality and quadratic generation. On the algebraic geometric side, on the first day Hal Schenck offered a toric tutorial for those coming from the polyhedral side. One group met to discuss approaches coming from adjunction theory, and there were tutorials on Frobenius splitting and on equivariant multiplicities. Yet another group met several days to discuss whether there could be an invariant, analogous to the Dehn invariant, that is additive on decompositions and zero on unimodular triangulations. Finally, one group met every day with the goal to create a census of three-dimensional smooth polytopes with few lattice points.

Immediate results. Several results were proved during the workshop.

(1) There is only a finite number of lattice d-polytopes containing fewer than k lattice points when the normal cones are restricted to a finite family of unimodular equivalence classes. (In the smooth case, this "family" contains only one element.) [Santos et al.]
(2) The set $C_{UT}(3)$ of all $k \in \mathbb{N}$ so that for every lattice polytope $P \subset \mathbb{R}^3$ the polytope kP has a unimodular triangulation contains $\mathbb{N} \setminus \{1, 2, 3, 5, 7, 11\}$. [Santos]
(3) Let D be a unimodular invariant of lattice 3-polytopes with values in an abelian group. If D is additive on subdivisions into lattice polytopes, then $D(P) = \mathrm{Vol}(P) \cdot D_0$ where D_0 is the value of D on the standard tetrahedron, and $\mathrm{Vol}(P)$ is the normalized volume of P.

    In this sense, there is no interesting $\mathrm{SL}(\mathbb{Z})$ analogue of the usual Dehn invariant. Up to torsion, this observation extends to arbitrary dimensions. [Fulton, Haase, Howard et al.]

Outlook. Several of the working groups plan to continue working on these problems. We now list some of these.

(1) Creating a census of smooth polytopes with few lattice points.
(2) Constructions of smooth polytopes in large dimension.
(3) Positive list of special polytopes.
(4) Studying truncated Nakajima polytopes.
(5) Studying centrally symmetric reflexive polytopes of high dimension.
(6) Unimodular triangulations in dimension four.
(7) Software improvements.
(8) Scheme-theoretic White conjecture.

We plan to record the progress on these questions at the projective normality website `http://www.warwick.ac.uk/staff/D.Maclagan/ProjNormal.html`

On a lighter note, after the banquet at the Hunan Gardens ten of the participants merged their six lucky numbers to form a six-dimensional polytope with nine vertices (two sets of numbers coincided). This “fortune” polytope turned out to form a convex polytope with almost palindromic h^*-vector, and with the unusual property that it has a large Hilbert basis with elements of the highest possible degree (in this case 5); in other words, the fortune polytope is as far from normal as it can get.