Trisections and low-dimensional topology

organized by
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Workshop Summary

This workshop gathered experts in various facets of 3– and 4–dimensional topology in order to better understand the interaction of trisections with other aspects of low-dimensional topology. Additionally, researchers worked to further develop the emerging theory of trisections, as many interesting foundational questions remain open. A trisection of a smooth 4–manifold X, introduced by Gay and Kirby in 2012, is a decomposition of X into three simple pieces, mirroring a Heegaard splitting of a 3-manifold.

In the first morning of the workshop, three of the four organizers gave talks in order to bring the audience up to speed in terms of the state-of-the-art in the field. The first afternoon consisted of a problem session, in which the entire group contributed a variety of interesting questions to examine over the course of the week and beyond. Akram Alishahi recorded and edited the problems so that they could be widely distributed.

In the remaining afternoons, participants split themselves into roughly eight small groups each day to work on specific problems. Membership in groups was fluid; some individuals stayed in the same group for the entire week, and others migrated between groups. The focus of some groups also fluctuated over the course of the week. Here we summarize the focus and progress of these groups:

Heegaard Floer homology: This group studied the interaction of Heegaard Floer homology with trisections, considering the question of whether there is a Floer-theoretic invariant that can be defined using trisection diagrams. Members made considerable progress toward the definition of a 4-manifold invariant and will continue to collaborate long after the workshop. The group had five or six members, depending on the day. It was important to understand both holomorphic triangle and quadrilateral counts, and to think about the closed case and the case with boundary. The group defined an invariant, but whether it is computable and/or interesting remains to be seen; its structure suggests it should be the same as the standard Heegaard-Floer 4-manifold invariant. The group is particularly interested in pushing beyond this to understand relative invariants and invariants of 4-manifolds which don't require b_2^+ to be positive.

Computing and trisections: This group looked at the extent to which computer programs could understand and depict trisections, asking questions about how best to encode 4-manifolds as data. They were extremely successful; at the end of the week, they had developed the framework for a program that takes a triangulation of a smooth 4–manifold as input and outputs a trisection diagram for that 4–manifold. This group had three members and will likely continue to collaborate beyond the workshop.

Generalizations of trisections: This group aimed to take a highly developed notion in 3–manifold topology, the theory of generalized Heegaard splittings and thin position, and adapt it to the realm of trisections. The group developed a working definition of a generalized trisection, which parallels ideas from its 3–dimensional counterpart. It remains open whether this definition can be transformed into a powerful theorem corresponding to the Scharlemann-Thompson thin position machinery for 3–manifolds. Membership ranged from three to five people, depending on the day.

Bridge trisections and invariants: This group was unique in that it was made up of three graduate students. A bridge trisection is a relative notion of a trisection for knotted surfaces in the 4-sphere, and group members successfully defined a new knotted surface invariant using the diagrams coming from bridge trisections, related to the idea of knot colorability in dimensions three and four. The students developed a computer program to compute the invariant. This group will continue to collaborate beyond the workshop, and the work they completed will likely result in a publication.

Symplectic and contact topology interactions: This group was initiated on the third day of the workshop, addressing the problem of understanding the ways in which symplectic and contact topology interact with trisections. Evidence of such a connection is provided by the fact that a trisection of a four-manifold with boundary naturally induces open-book decompositions of the boundary components, which are naturally related to contact structures. This group took a slightly different approach and focussed on understanding bridge trisections of knotted surfaces in $\mathbb{CP}^2$. Surprising success was made adapting the theory of bridge trisections to this important geometric setting.

Non-standard trisections of S^4: A major conjecture within the field of trisections is that all trisections of the 4-sphere are standard; that is, they can be constructed by generically modifying the simplest trisection of S^4. This group formed on Wednesday and attacked the problem from multiple angles, both attempting to prove the conjecture one day and trying to disprove it the next. Group membership varied, including some experts who had put a lot of thought into the problem and others who were thinking about it for the first time. One success of this group was the dissemination of existing (but unpublished) possible counterexamples. Membership in this group hovered around six.

Group trisections: Trisections naturally give rise to a similar decomposition of groups, defined by Aaron Abrams, Gay, and Kirby, called *group trisections*. This newly introduced decomposition is not well-understood, and the group worked on fundamental problems in group trisections. The group met for two days and had two or three members.

Homological data from trisection diagrams: This group included several individuals who had independently discovered how to use a trisection diagram of a 4–manifold X to compute the homology groups of X. They verified their individual approaches and discussed the best way to write down the details of these computations in order to make them more broadly accessible to the general research community. The three members of the group met for two of the four days, and they developed a plan to record and distribute their work.

Manipulating trisection diagrams: Group members looked at a particular set of examples, given by either the spin of a 3-manifold Y or the product manifold $Y \times S^1$. They drew and simplified a number of diagrams, eventually finding minimal genus diagrams for a large family of examples. This group met for two days and had three to four members. Results may be included in existing work in progress of one of the group members.

Computing L-invariants: During a Tuesday morning talk, Abby Thompson presented joint work with Gay and Kirby which defines a new 4–manifold invariant from trisections which takes values in the nonnegative integers. This group met to undertake calculating the invariant for a class of low-complexity examples and to address open problems, such finding the smallest non-trivial L-invariant and corresponding minimal 4–manifolds. This group met for a couple days and had two to three members.

The mapping class group of a trisection: The mapping class group of a given trisection of a 4–manifold X is the group of diffeomorphisms of X preserving the trisection, considered up to isotopy. This problem has been studied for Heegaard splittings of 3-manifolds; for S^3, a classification of its generators is the well-known Goeritz Conjecture. This group addressed the basic questions of the same problem in dimension four. It included two or three members and met for two days.

One high point of the workshop was a collection of three talks given by graduate students on Thursday morning, each about previous trisection-related research. One of the talks led to the the creation of the small group that manipulated trisection diagrams.

The workshop culminated in a wrap-up session on Friday afternoon, in which various small group members presented the results of their work to the rest of the conference participants. The large group also discussed interesting problems stemming from this work and future research directions. The workshop will very likely result in a publication for the group of graduate students who introduced colorability of bridge trisection diagrams. Additional collaborations/publications are possible from the Heegaard Floer homology group, the computing and trisections group, and the symplectic topology group, who indicated their intention to apply for an AIM SQUARE.

Overall, we believe that a decade from now, this AIM workshop will be viewed as a turning point in the maturation of this field, as well as a catalyst for a far greater understanding and appreciation of trisections as a prominent tool in 4–manifold topology.