Low dimensional structures in dynamical systems with variable time lags

organized by
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Workshop Summary

The workshop discussed differential equations with time lags which are state-dependent. Such equations are not covered by the standard theory of functional differential equations, as it is presented in monographs which appeared during the past few decades: see [2, 1]. The reason for this deficiency arises from a lack of smoothness of the delay as a function of the state and its history. Motivation to study differential equations with state-dependent time lags comes from numerous models in science and technology. While it is often agreed that delays when present in a dynamical system are state-dependent, the specific form of state-dependence remains hard to describe in many cases.

We had 21 participants (18 invitees and 3 applicants). They came from North America (USA 8, Canada 3), Latin America (Chile 1), and Europe (EU 8, Switzerland 1), among them 2 graduate students and 4 postdocs. There were experts in the mathematical analysis of differential equations with state-dependent delay, newcomers (about a third of the participants), and experts in numerical analysis and modelling.

Each morning started with 2 introductory lectures, followed in the afternoon by shorter presentations and group discussions, which became more focussed and intense in the course of the week. Friday afternoon we collected a list of open problems which we believe to be crucial for further progress.

Monday began with an introduction by Walther into existence theory for the initial value problem on a Banach manifold which is defined by the equation considered. This approach yields a semiflow of differentiable solution operators, and tools from dynamical systems theory become applicable. Also addressed were the concept of deferred action, which motivates the study of algebraic-delay systems with the delay given by an algebraic equation, the phenomenon of reactions in reversed temporal order, whose mathematical description relies on *variable* time lags, and Poisson's first paper on differential equations with state-dependent time lags, from 1806 [8].

The second lecture, by Krisztin, introduced the construction of continuously differentiable local invariant manifolds. In the case of differential equations with state-dependent delay this is considerably more involved than for other types of differential equations. Higher order differentiability of the finite-dimensional local center and unstable invariant manifolds requires a set of hypotheses which is specific for state-dependent delay. For stable and other *infinite-dimensional* invariant manifolds higher order differentiability is unknown.

The afternoon session consisted of several shorter contributions, embedded in a discussion in which we tried to identify themes for informal working groups.

Mallet-Paret started with a brief introduction to the results [4, 5, 6] obtained jointly with Nussbaum concerning the asymptotic shape, uniqueness and stability of periodic solutions for a prototypic class of nonlinear autonomous equations with a small parameter. Here the asymptotic shape of a periodic solution is a set in the plane, and an important tool to find it is the *backdating map* in dimension 2. Mallet-Paret then discussed related open problems, concerning the form of the delay and the possibility of extensions to equations with several delays.

Barbarossa described her project on population models, which include equations of neutral type and unbounded delay. For neutral equations with state-dependent delay the basic theory along the lines of the first lecture is more challenging and thus less developed, with smoothness results weaker. In particular, linearization at equilibria is still an issue, and existence of local invariant manifolds is an open problem.

Stumpf presented his recent work about an equation modelling a price. He established existence of a 2-dimensional global center-unstable manifold with a periodic orbit as the manifold boundary.

Kennedy explained his results on multiple periodic solutions for equations with nonlinearities close to step functions. The proofs begin with simplified equations whose dynamics are given by a collection of maps in finite-dimensional spaces. Kennedy also described objectives of further research.

The lectures on Tuesday and Wednesday morning dealt with numerical results, modelling and simulation. On Tuesday Humphries began with numerics. He presented results obtained with DDEBIFTOOL for equations with 1 and 2 delays, among them global Hopf bifurcation and bifurcation to invariant tori. The latter seems still out of reach of mathematical analysis.

In the next lecture Erneux explained models for chatter instabilities during deep drilling - related to the recent Deep Water Horizon disaster in the Gulf of Mexico. Other models with state-dependent delay concerned coupled lasers, malaria infection, and in greater detail, car following. The lecture also showed how to gain first insight into solution behaviour by means of heuristic use of asymptotic methods involving multiple time scales.

On Tuesday afternoon we had a further plenary session with short presentations. Walther reproduced the derivation of the equation studied by Poisson from a problem in planar geometry and suggested further work on this equation, which is not covered by present day theory since the delay involves the derivative of the solution and there are advanced arguments.

Landsman explained a model for 3 lasers, with constant delays in the coupling, and Calleja reported about his work in KAM theory.

Then we split into groups on car-following models, smoothness issues (with analyticity of periodic solutions), and population models as in Barbarossa's brief talk, and began to discuss these topics.

Wednesday morning Krauskopf spoke about delays in hybrid testing and lasers, with numerical results for some models, and discussed the nature of the delays and their relevance.

In a shorter presentation Gedeon explained gene regulatory processes in cells, with a focus on the different delays arising in transcription and transport processes.

In the afternoon we had discussions in the groups formed the day before. In the smoothness group Krisztin presented his extension of Nussbaum's result on analyticity of periodic solutions [7] to some equations with state-dependent delay.

Thursday morning and Friday morning were again devoted to mathematical analysis. On Thursday Hartung explained his recent results on differentiability of solutions with respect to parameters. This includes smoothness with respect to initial data, for the solution value in an Euclidean space and also for the solution segment in some history spaces. Such results are relevant for optimization and parameter identification problems.

Then Hupkes spoke about his work on finite-dimensional center manifolds for equations of mixed type with constant time lags and advancements. The associated initial value problem is ill-posed in general. Applications occur, for example, in modelling signal propagation along nerve fibres and in the search for travelling waves in lattice dynamical systems. Equations of mixed type with state-dependent delays and advancements arise in the N-body problem of electrodynamics; at present this area largely is terra incognita.

The group discussions on Thursday afternoon included short presentations by Hoffman on travelling waves in unidirectional lattice dynamical systems and by Yuan on bifurcation with symmetries for a model of a ring of neurons, with constant delay in the connection terms.

In a further group we began to discuss the work of Stumpf, with the aim to relax a condition on the delay which prevents the study of the price model as a singular perturbation problem, similar to the approach of Mallet-Paret and Nussbaum [4, 5, 6].

The first lecture on Friday morning, by Magal, addressed Hopf bifurcation for non-densely defined problems. These include partial functional differential equations (with constant lags). The tool used are integrated semigroups.

Then Trofimchuk gave a shorter lecture on equations with maxima, which involve the maximum of the solution in a certain history interval. Such equations are motivated by control problems. They obviously exhibit a specific lack of smoothness. The lecture explained how the dynamics generated by an equation of this kind can be reduced to that of a system in one dimension, which is given by a non-monotone interval map. This leads to existence of chaotic solution behaviour, among other phenomena.

Following suggestions made after Trofimchuk's talk, the last afternoon began with a short lecture by Nussbaum on max-plus equations (which are not differential equations but contain transformed arguments). A theory of max-plus equations and a related eigenvalue problem was developed by Mallet-Paret and Nussbaum as a tool which helps to identify limit profiles of periodic solutions in [4, 5, 6]. Friday's lecture dealt with spectral radius results.

Next we collected problems, in a session chaired by Krauskopf. We tried to include what had been understood as a major issue during the workshop. The resulting list, formulated by Stumpf, will be posted on the website along with this report.

The remaining time was used for discussions. In the small group started on Thursday we developped a plan for proving existence of Stumpf's periodic solution without his restrictive hypothesis on the delay. The method of proof would be different, without a global invariant manifold. We hope that this meeting was the beginning of joint work on different parts of the plan of attack.

The workshop also initiated further collaborations. Krauskopf and Stumpf plan to work on numerical approximation and visualization of global invariant manifolds as in Stumpf's work, also in cases where nothing could yet be proved. Kennedy and Stumpf as well as Calleja and Krauskopf intend to collaborate. Erneux, Mallet-Paret and Nussbaum had a preliminary discussion about joint work on a two-parameter problem related to the sawtooth profile, which occurs in one of the guiding examples in [4, 5, 6]. Trofimchuk and Walther may begin to study Poisson's state-dependent delay equation from a dynamical systems point of view. Krisztin and Walther discussed the need for an introductory textbook about delay differential equations which would make the field more easily accessible to students.

The group of participants was small. In light of this there was still a great deal of heterogeneity. The workshop brought into contact various groups which had previously had little or no interaction - young researchers and senior experts, various subgroups of young researchers, and colleagues working on widely separated parts of the field, such as modelling and pure mathematics. This last aspect seems important, as differential equations with state-dependent delay require somewhat subtle mathematics which model builders should be aware of, while an understanding of the role of delays in applications may guide mathematical research in the jungle of possible relations between state histories and delays caused by these histories.

It is our hope that the workshop led some of the participants to a deeper interest in the subject and will encourage them to begin to work in this area, which is to a large extent still unexplored and full of challenging problems.

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