Vector equilibrium problems and their applications to random matrix models

organized by
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Workshop Summary

This AIM workshop focused on the use of vector equilibrium problems in random matrix theory. The workshop was able to collect a good mix of experienced senior people and younger early career mathematicians.

The main topic of the workshop was the analysis of random matrix models with the use of vector equilibrium problems and other novel methods. It is a classical result that the eigenvalues of invariant random matrix ensembles are asymptotically distributed according to the equilibrium measure in an external field. Their analysis is intimately related to orthogonal polynomials, integrable systems and Riemann-Hilbert problems.

In recent years a number of matrix models were connected to multiple orthogonal polynomials. This is a notion that originates from the theory of Hermite-Padé rational approximation and vector equilibrium problems are used to describe the convergence of rational approximants. These ideas turn out to be very useful for the analysis of random matrix models as well. In particular vector equilibrium problems have been used successfully for random matrices with external source, two matrix models and the normal matrix model.

A next step in the application of ideas from Hermite-Padé approximation to random matrices would be to incorporate the notion of S-curve in a vector equilibrium problem. This notion arises when there is no a priori knowledge about the position of an appropriate contour. In a random matrix model, this could apply to the supports of the secondary measures in a vector equilibrium problem that describes the limiting eigenvalue distributions.

These issues were discussed at length in the AIM workshop. In the morning sessions there were interesting presentations which in the afternoons were followed by full group discussions and discussions in smaller groups to which all of the participants contributed. On Tuesday afternoon there was an open problem session in which 15 challenging open problems were posed by various participants. The discussions in smaller groups focused on specific topics. In the present report we review the main issues that were discussed during the workshop, and we formulate some fundamental unsolved problems for the future studies.

S-curves.

A major development in potential theory is Rakhmanov's recent proof for the existence of S-curves in harmonic external fields, based on the consideration of a max-min energy problems. The proof also relies on earlier work with Martínez-Finkelshtein on the regularity of critical trajectories of quadratic differentials.

Rakhmanov and Martínez-Finkelshtein presented their work in plenary talks, that were followed by more detailed explanations of the proof in a smaller group. Similar issues can be expected to arise in a vector setting, and therefore these discussions were highly instructive.

One group explored the possible behaviors of S-curves in the case of piecewise harmonic external fields, in particular for a hat like external field.

External source model.

A serious restriction in recent work is the assumption of enough symmetry so that the equilibrium measures from a vector equilibrium problem are a priori known to be supported on the real or imaginary axis.

This is valid in particular for the work of Bleher, Delvaux and Kuijlaars on the random matrix model with external source, where the potential is assumed to be an even polynomial, and the external source is restricted to having only two eigenvalues with equal multiplicities.

A discussion group met several times during the week to discuss this issue and to explore the possibilities of lifting the symmetry restrictions. Progress was made on calculations around a simple toy model with a quadratic potential. An alternative approach was outlined by Lysov.

Problem 1. *Develop the Riemann-Hilbert approach to the external source model with a general, not necessarily even, potential function, and obtain in this way the universal generating kernels for correlation functions of the external source model in the bulk, at the end-points, and in the double scaling limit at the critical points.*

This problem requires further development of the S-curves and the general theory of the matrix-valued Riemann-Hilbert problem in applications to the scaling limits and double scaling limits in the ensembles of random matrices with external source.

Two matrix model.

The two matrix model is another model where vector equilibrium problems appear. Duits presented the vector equilibrium approach from his work with Kuijlaars and Mo, while Bertola explained his recent work with Gekhtman and Szmigielski on the related Cauchy two matrix model.

A group discussed a possible extension of the vector equilibrium to a two matrix model with certain quartic potentials. The calculations were found to be quite involved. However the implications of a successful analysis will be very interesting, and so the topic is worth pursuing in more detail in future work.

Problem 2. *Develop the Riemann-Hilbert approach to a general two-matrix model, and more generally, to a chain of interacting random matrices.*

This is a fundamental problem of the theory of interacting random matrices, and it can have important far-going applications to the theory of quantum gravity and statistical mechanics on random surfaces.

Normal matrix models.

Lee and Makarov gave stimulating presentations on their work on the normal matrix model. Lee in his lecture described his join work with Balogh, Bertola, and McLaughlin devoted the normal model with logarithmic perturbation of the quadric potential. An insight

from this important work is that in certain cases the planar orthogonality that is associated with the normal matrix model can be replaced by orthogonality on contours. This is a very important step towards asymptotic analysis since Riemann-Hilbert methods become available. Another important recent achievement is the paper of Bleher and Kuijlaars where the Riemann-Hilbert approach was successfully applied to the contour orthogonal polynomials associated with the cubic normal random matrix model.

A discussion group met several times during the week to explore the implications and possible extensions of the Balogh-Bertola-Lee-McLaughlin approach and its relation to the Bleher-Kuijlaars method. The group tried to implement the approach to the important cases of the harmonic polynomial perturbations of the quadric potential. It turns out that indeed there is, in principal, a mechanism of transforming the original planar orthogonality into the orthogonality on contours in this case as well. This important observation has yielded two interesting preliminary results. In the case of the cubic perturbation of the quadric potential, the calculations performed by the group produced, module still unresolved cut-off issue, the 3×3 Riemann-Hilbert problem which plays the key role in the Bleher-Kuijlaars formalism. The second preliminary result is concerned with the cubic perturbation of the quartic potential. In this case there is no need for any cut-off, and the group's calculations yielded a Riemann-Hilbert setting in a completely rigorous way.

An interpretation of the normal matrix model in terms of spectral determinants was discussed in another group. Wiegmann contributed with a clear exposition and many inspiring insights.

Problem 3. *Develop the Riemann-Hilbert approach to the general normal random matrix models with polynomial interaction. Investigate bifurcations and double scaling limits at the critical points of the normal random matrix models and the related Laplacian growth model.*

The above mentioned preliminary calculations performed during the AIM workshop are very likely to become a starting point in this analysis.

Spectral curve approach.
The spectral curve approach to equilibrium measures is based on the utilization of an a prior information available via underlying Lax pairs. The basic idea is to identify the multi-valued algebraic functions representing equilibrium measures with the algebraic curves determined by WKB - characteristic equations generated by the corresponding Lax pairs. This point of view has already proven its usefulness in a number of concrete examples. One of the discussion groups investigated the future prospects of this approach. In particular, the algebraic curve describing the equilibrium measure found by Bleher and Kuijlaars for the cubic normal matrix model was considered. It was observed that the Bleher-Kuijlaars curve coincides indeed with the spectral curve formally derived from the associated 3×3 linear differential system.

β ensembles.

Shcherbina gave a very interesting presentation of the state of art in the theory of general β - ensembles. She reviewed the recent work of Borot and Guionnet where the complete asymptotic expansion was found in the one-cut case and underlined that difficulties

which the Borot-Guionnet method is facing in the multi-cut case. She also described her important recent work on expansions for β ensembles which was based on the so-called loop equation. The talk was met with much interest and discussion. It was followed by a small group discussion on a technical issue that came up during the talk. In fact, it was noticed that in both approaches a special integral operator belonging to the distinguished "integrable class" is appearing. This suggests that an algebrogeometric scheme based on the theory of integrable Fredholm operators can be in principal applied to the multi-cut case.

Problem 4. *Obtain the asymptotic expansion of the partition function and correlation functions in the β ensembles with multicut equilibrium measures.*

This problem is a basic, fundamental problem and its solution would be instrumental in various applications. It is conceivable that the observed during the AIM workshop connection to the integrable Fredholm operators would help in attacking the problem.

The workshop at AIM took place in a very friendly and enjoyable atmosphere that was appreciated very much by all of the participants. The staff at AIM were of great help and contributed very much to the smooth organization of the meeting. The workshop will no doubt contribute to the further development of new ideas in random matrix theory, equilibrium problems and asymptotic analysis of Riemann-Hilbert problems.