Wall-crossing: techniques and applications

organized by
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Workshop Summary

The main focus of this workshop was to disseminate and collaborate on recent developments in wall-crossing techniques, and to identify opportunities where they can be applied. This included the recent "universal" wall-crossing machinery of Joyce as well as the non-abelian localization techniques of Halpern-Leistner, and potential applications included enumerative problems for sheaves on Calabi–Yau 4-folds, Bridgeland-stable objects in triangulated categories, and more.

We had two talks every morning (Monday–Friday) on a wide range of topics that touched on almost all wall-crossing techniques which exist in the current literature. We chose to ask some participants to give talks reviewing already-established wall-crossing frameworks, e.g. that of Joyce–Song and Kontsevich–Soibelman for motivic Donaldson–Thomas-type invariants, whose previous applications should be re-examined and refined in light of the newer Joyce-style wall-crossing machinery for enumerative invariants built from virtual cycles and their equivariant and/or K-theoretic versions.

The Monday afternoon problem session was particularly productive, with enthusiastic and productive input from almost all participants, resulting in a list of nine broad but actionable directions of possible research. Four out of these nine directions had active discussion groups throughout the week, and some of these groups plan to continue to meet in the future. Many of these directions contained multiple sub-problems, to be stated in a list of open problems which we believe will be extremely valuable for the enumerative geometry community.

Overall, the organizers were extremely pleased by the dynamic environment of the online workshop and the level of engagement of all the participants. We feel that the workshop was successful in bringing together a diverse group of both wall-crossing practitioners and experts in adjacent areas which will benefit from wall-crossing techniques, and sparking many in-depth discussions that will pave the road for future work and collaborations.

Modularity of partition functions.

This group discussed the possibility of using vertex algebra techniques to attack conjectures on the modularity of various partition functions, of sheaf-counting flavor, in the literature. One main focus was to identify some precise conjectural statements, e.g. in Vafa–Witten theory, and to clarify how they may be (un)related to each other. There was some preliminary discussion on using the (lattice) vertex algebra structure on the moduli stack of zero-dimensional sheaves, and, via point-modification operators, to make it act on the desired moduli spaces of (semi)stable objects. Some concerns were raised about the difficulties of studying (the representation theory of) lattice vertex algebras associated to *non-positive-definite* lattices, indicating some promising future directions to pursue.

Non-abelian localization techniques.

The group started on Tuesday by reviewing the existing non-abelian localization formula for K-theory by Dan Halpern-Leistner; in particular, how it can be used in the case of quasi-smooth schemes. The rest of the days, we tried to understand whether the approach could be applied to the case of the Oh–Thomas class on −2-shifted symplectic stacks. Based on the torus localization formula proved by Oh–Thomas, it seemed evident that some analogue should hold. A key question was whether the same formula (applying exceptional pullback to the stratum, followed by pushforward to its center) would give the right results, or whether the exceptional pullback operation would need to be modified. We found that this recovers the expected formula in the simple example of a −2-shifted cotangent bundle over the affine line (with its natural torus action). However, this is a very special case. We started the calculation in a more general example that doesn't reduce to the quasi-smooth case, but hadn't gotten to a conclusive answer yet. This presents an interesting starting point for further investigations.

We also briefly discussed the −1-shifted symplectic setting, but found it harder to approach due to the lack of a virtual structure sheaf on Artin stacks with −1-shifted symplectic structure.

(Combinatorial) problems arising from 4*-folds.*

The group met every afternoon from Tuesday to Friday and discussed potential combinatorial approaches to proving certain closed-form formulas arising from Nekrasov's magnificent four formula (now a theorem of Kool and Rennemo). We focused mainly on the identity

$$\sum_{n\in\mathbb{Z}} q^n \int_{[\mathrm{Hilb}^n(\mathbb{C}^4)]^{\mathrm{vir}}} 1 = \exp\left(Kq\right) \tag{1}$$

where K is an explicit rational function of the equivariant variables, in particular independent of q. We discussed how the form of the right hand side makes it particularly amenable to an inductive proof using branching rules, similar to how one proves the hook-length formula for Young tableaux, and concluded that this is a promising approach to a purely combinatorial proof of (1).

Wall-crossing for triangulated categories.

This group met on Tuesday, Wednesday, and Thursday in the afternoon and discussed possible extensions of Joyce's wall-crossing formula to the case of Bridgeland stability conditions on triangulated categories. We spent time trying to determine some good cases to try to understand what an extension of Joyce's formula should say. After consulting the literature for some time, we settled on investigating some examples related to representations of quivers. We considered the simplest non-trivial example, the bounded derived category of the projective line. The interesting behavior is that this triangulated category has stability conditions coming from King stability on the 2-Kronecker quiver and stability conditions arising from slope stability. We spent some time working out what happens upon crossing a wall where the heart undergoes a transformation, since the other types of walls already are a part of Joyce's framework. We believe we figured out that these types of walls do not present new phenomena in this case, so at least at present it seems plausible that one can apply a version of the Abelian wall-crossing formula to paths in the space of stability conditions on $\mathbb{P}^1$. There are some plans to continue discussing this example and possible generalizations of the observations.