Mathematical questions in wave turbulence theory

organized by
Tristan Buckmaster, Pierre Germain, Zaher Hani, and Jalal Shatah

Workshop Summary

The goal of the workshop was to bring together experts from physics and several fields of mathematics in order to advance the mathematical understanding of wave turbulence theory. Wave turbulence is a branch of equilibrium statistical physics that tries to describe the effective dynamics of nonlinear waves over long time intervals, or in the presence of a random forcing. It was developed mostly by physicists and applied mathematicians in the past century, but very little is known rigorously in this field.

A central emphasis was placed on the validity and possible justification of the so-called wave kinetic equation. This equation is the wave analog of Boltzmann's equation for particle interactions. Like Boltzmann's equation, its derivation is a major mathematical question that is still open. Several talks and problem sessions were aimed at taking steps to better understand this problem.

1. Talks

The workshop started by two talks on the physics and applied mathematics perspective by Sergey Nazarenko and Themis Sapsis, who described the relevance relevance of wave turbulence from the applied perspective. On later days, the talks were more mathematical: Laure Saint-Raymond discussed several challenges that make the problem of the derivation of the wave kinetic equation a difficult one; Patrick Gerard discussed a mathematical problem very much related to wave turbulence, namely growth of Sobolev norms, particularly in the context of the Szegö equation; Tristan Buckmaster discussed a recent joint work in progress with the organizers that takes a step in the derivation of the wave kinetic equation; Minh Binh Tran discussed the mathematics of the wave kinetic equation from the well-posedness point of view; Daniel Tataru discussed the derivation of conserved quantities for some integrable equations; Alex Ionescu discussed stability questions for some quasilinear dispersive equations; and finally Gigliola Staffilani discussed some probabilistic tools in the analysis of nonlinear partial differential equations. All speakers proposed open problems, some of which were discussed in the working groups.

2. Working groups

The working groups discussed the following problems:

(1) Derivation of the wave kinetic equations and its relation to similar problems, like the justification of Boltzmann's kinetic equation.
(2) The phase randomness assumption of wave turbulence.
(3) Well-posedness problems for the wave kinetic equation.

(4) Well-posedness thresholds for the MMT equation.
(5) Random data for the Szegö equation.
(6) Well-posedness problems for the SQG equation.

The discussions led to an improved understanding of the problems involved, particularly as they featured an exchange of expertise between participants from different research backgrounds. Several of them led to continued discussions after the workshop in the form of collaboration projects.