Zero forcing and its applications

organized by
Shaun Fallat, Simone Severini, and Michael Young

Workshop Summary

The workshop "Zero Forcing and its Applications met at AIM January 30, 2017 - February 3, 2017. The workshop was on the graph propagation Zero Forcing, its many application, and its variants. There were nine talks given:

- Leslie Hogben: Introduction to Zero Forcing
- Bryan Shader: Relationship to Linear Algebra
- Daniel Burgarth: Relationship to Quantum Control
- Daniela Ferraro: Relationship to Power Domination
- Franklin Kenter: Computational Zero Forcing
- Jephian Lin: Variations of Zero Forcing
- Tracy Hall: q-Zero Forcing
- Boting Yang: Fast-Mixed Searching
- Steve Butler: Propagation Time

During the problem session on Monday afternoon, the workshop participants amassed a list of over 30 potential problems. Over the course of the week the workshop participants primarily focused on six different problem formulations.

Group 1: Wayne Barrett, Steve Butler, Minnie Catral, Craig Erickson, Shaun Fallat, Tracy Hall, Brenda Kroschel, Jephian C.-H. Lin, Bryan Shader, Nathan Warnberg, and Boting Yang.

Computations concerning q-zero forcing

The notion of q-zero forcing offers an extension of both the original zero forcing game and positive zero forcing. Furthermore, the q-zero forcing number of a graph G (Z_q) is a combinatorial upper bound on the maximum nullity over all symmetric matrices with graph G and having exactly q negative eigenvalues. One of the prime objectives of the group was to characterize the graphs with extreme values of Z_1 and Z_2.

In addition, it was previously known that the original definition of q-zero forcing has some unfortunate limitations, they attempted to address some of these limitations in the context of the inverse inertia problem for trees. During the week our group characterized the graphs in which $Z_1 = 2$, and based in part on this analysis they established the following general facts:

1) For any graph G, if $Z_k(G) \leq k$, then $Z_k(G) = Z(G)$.

2) For any graph G, if $Z_k(G) = k$, then $Z_{k-1}(G) = k$. Furthermore, they showed that there exist a class of graphs for which $Z_k(G) = k + 1$, but $Z(G) = k + 2$. Turning to binary trees, they demonstrated that the gap between Z_1 and M_1 (the maximum nullity as

described above with $q = 1$) can be arbitrarily large. Finally, they started to address the limitation of the current definition of q-zero forcing, and devised an improved zero forcing rule. This newly developed rule was made specific for Z_1, and labeled as $\tilde{Z}_1$. This improved 1-zero forcing did prove to be very interesting, and in particular, for certain binary trees T they proved that $\tilde{Z}_1(T) = M_1(T) = 2$, whereas they have that $Z_1(T) = i$, where i is the depth of T. Moving forward, they are planning on writing and publishing our preliminary report on extreme values of Z_q, plateaus in q-zero forcing, and on the improved Z_1 forcing game.

Group 2: Ellen Gethner, Cheryl Grood, Daniella Ferraro, and Ruth Haas.

Zero Forcing and the Art Gallery Problem

Inspired by the result that an upper bound for the zero forcing number of a planar graph on n vertices is $n/3$, and the fact that any simple polygonal region with n vertices is visible by no more than $n/3$ guards (the Art Gallery Theorem), the group delved into the proof of the Art Gallery Theorem to see if the technique or at least the spirit of the technique would carry over to zero forcing problems. The aim is to find the exact zero forcing number of any edge-maximal planar graph. The group's investigations led them to a related problem, which they solved completely. The problem is: take the infinitely family of graphs given by the recursively defined Sierpinski triangles (based on the Sierpinski Sieve) and find the zero-forcing number of each. This was done by utilizing the recursive nature of the construction together with several case-by-case arguments. The group then moved on to the Sierpinski Carpet graphs, which although similar in nature, are proving to be a challenge. They developed enough insight that it is believed that future work should lead to an exact solution, as well.

The group plans to continue to meet to go after the larger problem of finding the exact forcing number of edge maximal planar graphs; the first such meeting will likely take place at CanaDAM in June 2017 and lead to meetings at their various institutions over the coming years. The group anticipates they will find other kinds of problems to work on together.

Group 3: Chassidy Bozeman, Boris Brimkov, Craig Erickson, Daniella Ferrero, Mary Flagg, and Leslie Hogben.

Restrained Zero Forcing

Electric networks are frequently modified, sometimes by building an extension with new nodes and lines. The labor to install a PMU and the supporting non-portable infrastructure are significant parts of the total cost of installing a PMU at a given node. Thus, it is a problem of interest to determine the minimum number of additional PMUs needed, and the locations where they should be placed, when an existing network is expanded and the existing PMUs remain in place. The group considered the problem of finding a power dominating set that contains a given set of vertices S and minimizes the total number of PMUs used subject to the constraint that the vertices in S are included. They are also studying the related problem of finding zero forcing sets that are required to contain a given set S of vertices. Such power dominating and zero forcing sets are called "restrained" with respect to S.

During the workshop and in the following weeks, the group came up with several tight bounds on the restrained power domination numbers and zero forcing numbers in terms of their non-restrained analogues and other graph parameters. The theory developed also supports algorithmic results; in particular, they have found a linear time algorithm for the restrained power domination number of a graph with bounded tree-width, given an arbitrary restraining set S. They are also devising a parallel algorithm for the power domination number of trees, where the tree is partitioned into appropriately constructed smaller trees whose power dominating sets can be computed independently. Future work will focus on extending the algorithms to more general graphs, and deriving other bounds on the restrained power domination and zero forcing numbers.

Group 4: Louis Deaett, Illya Hicks, Ruth Haas, and David Roberson

Forts in Graphs

A fort in a graph is a subset F of vertices such that no vertex outside of F is adjacent to exactly one vertex of F. With respect to zero forcing, a fort is the set of vertices that do not get forced by some initial set of colored vertices, and a zero forcing set is exactly a set of vertices hitting every fort. This group focused on exploring theoretical aspects of forts with an aim to obtain better and/or more practical bounds on the maximum nullity, $M(G)$, of a graph G. The first thing they proved is that for any matrix A that fits a graph G, the support of any null vector of A is a fort of G. Conversely, for any fort F of G and any vector x whose support is F, there is a matrix A that fits G with x as a null vector. These statements are true for any field except for $GF(2)$.

They also used the observation that, in order to bound $M(G)$, one can consider a single matrix fitting G at a time. What this means is that, instead of looking for a set of vertices hitting every fort of G, i.e., a zero forcing set, one can look for a set of vertices that hits the support of every null vector of a particular matrix A fitting G. The minimum size of such a set is equal to the nullity of A. Taking the maximum over all A that fits G gives $M(G)$. Of course we do not want to have to maximize over all A that fits G, but we can use this idea as follows: If A fits G, then the minimal supports of its null vectors are forts of G, and they form the circuits of a matroid (the matroid represented by the columns of A). Thus it is only needed to consider all subsets of forts that form the circuits of a matroid, find a minimum hitting set for each, then take the maximum cardinality of these hitting sets. This will give an upper bound on $M(G)$ which will hopefully be better than the zero forcing number. Unfortunately, this does not seem to be very efficient, as even small graphs can have very many forts. However, the group is interested in how good this bound is theoretically, and they plan to continue investigating the theoretical properties of forts with the aim of finding more efficient ways of computing this or perhaps other bounds on $M(G)$.

Group 5: Daniela Ferrero, Mary Flagg, Tracy Hall, Leslie Hogben, Jephian Lin, Seth Meyers, Shahla Nasserasr, and Bryan Shader.

Partial Zero Forcing

The basic idea of the problem that this group worked on is to take a set of paths that behave like zero forcing chains but which do not necessarily cover the entire graph, and see if there is a combinatorial relationship between the number of paths, the number of vertices in the paths, and the allowable multiplicities of a matrix for the graph.

The group was able to prove a result that if there are t paths that contain m vertices which satisfy a certain uniqueness condition (which is satisfied for the zero forcing chains of a zero forcing set), then the Ferrers diagram of the partition of $|V|$ given by the multiplicities of the eigenvalues contains at least m boxes in the first t columns. This connects two previous bounds: it gives the usual zero forcing result for maximum nullity since our result would say that all $|V|$ of the eigenvalues live in the first $Z(G)$ columns, or equivalently, that $M(G) \leq Z(G)$, and similarly, the length of the longest unique path is then a lower bound for the number of distinct eigenvalues. At the end of the workshop the group was working on a combinatorial modification of the usual zero forcing game which would generate the appropriate type of unique paths.

Many of the group's tasks were completed in the two weeks following the workshop, including writing up the proofs formally and finishing the details of the new zero forcing type game, which is being denoted by $Z_{\#}$. The group anticipates publishing a paper in the medium term.

Group 6: Michael Dairyko, Franklin Kenter, Karen Meagher, and Michael Young

k-power propagation in hypergraphs

Chang and Rousell conjectured that in an r-uniform hypergraph H with n vertices the k-power propagation number of H is bounded above by $\frac{n}{k+r}$. They proved that it is true for graphs ($r = 2$), and that $\frac{n}{k+2}$ is an upperbound for the k-power propagation number of any r-uniform hypergraph.

A group worked on proving this conjecture. While not completely proving it, the group has been able to prove that the conjecture is true when r is 3 or 4. They have also shown that $\frac{n}{k+4}$ is an upperbound for the k-power propagation number of any r-uniform hypergraph with $4 \leq r$ and that the conjecture is true for all r-uniform linear hypergraphs.

The group anticipates submitting a paper for publication some time this year.