Moments of zeta and correlations of divisor sums

organized by
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Workshop Summary

The AIM workshop "Moments of zeta and correlations of divisor sums" was held at the American Institute of Mathematics in San Jose, CA from Monday, August 29 to Friday, September 2, 2016. The workshop was organized by Siegfred Baluyot and Steve Gonek of the University of Rochester, and Jon Keating of the University of Bristol.

Despite their importance, rigorous proofs of asymptotic formulae for the moments of the Riemann zeta function and other L-functions are only known for a few cases. In the mid 90's, Conrey and Ghosh conjectured a formula for the sixth moment of zeta on the critical line and later Conrey and Gonek developed a heuristic method based on moments of long Dirichlet polynomials that gave the second, fourth, sixth, and eighth moments. The method failed, however, for the tenth moment. At the same time, J. P. Keating and N. Snaith used techniques from random matrix theory to conjecture an asymptotic formula for all the moments. In the years that followed, this conjecture was made more precise and similar conjectures have been made for other families of L-functions by heuristic methods on the number theory side, now known as "the recipe" and "the ratios conjecture".

The mystery behind the breakdown of the long Dirichlet polynomial method of Conrey and Gonek for high moments was never adequately explained. However, in a recent series of articles Conrey and Keating have revisited the issue and given an in-depth analysis of the long polynomial approach that reveals why it fails after the eighth moment, and how it may be corrected. It has emerged that there are neglected terms in this approach and that similar terms arise in a host of other problems such as in the variance of the divisor function in short intervals and in the variance of the divisor function in arithmetic progressions. It also turns out that the calculation of these terms is reminiscent of the circle method.

The workshop had several general goals. One was to apply the method to a wider range of problems. A second was to begin the work of making the method rigorous where possible. A third was to elucidate the connection between the circle method and the current method.

During each of the first four mornings, two lectures on relevant material were delivered. The two lectures on the first day described the method as it applies to the Riemann zeta function. A large list of specific problems was drawn up the first afternoon, and the organizers whittled this down to the following six:

(1) Apply the method to calculate the variance of divisor sums in short intervals.
(2) Work out analogues for L-functions in the function field setting.
(3) Try to prove the "recipe", which heuristically predicts all terms in moment calculations, for the fourth moment of the zeta function.

(4) Determine how the so-called Type I and Type II sums are related. More specifically, the former would initially seem to contain the latter. How does one see that the latter need to be separated out?
(5) Use the method to predict precise formulae for moments of L-functions in other families such as quadratic Dirichlet L-functions and cusp form L-functions.
(6) Carry out rigorous proofs for low moments in various families of L-functions assuming minimal realistic hypotheses.

The workshop participants split into groups consisting of three to nine people, according to their interests. Most of the groups made good progress through the week, which was reported daily to the entire workshop, and some members in them agreed to continue their work after the workshop. We now describe the work of two of the groups, those working on problems 2 and 4, because we feel it was particularly noteworthy.

The focus of those working on problem 2 was to calculate the second moment of L-functions in the hyperelliptic ensemble over function fields. To begin with, it was not at all clear what the analogue of Type I sums should be in the context of quadratic character averages. This was eventually sorted out by the group, and it turned out that, roughly, square polynomials correspond to the "diagonal", and squares times square-frees correspond to the off diagonal Type I terms. The group will continue working on the second moment calculation after the workshop. We consider their progress substantial because it shows the way forward to applying the method in a next family of L-functions. It seems clear that the ideas here will also work for quadratic L-functions over the rational function field too (problem 5).

Perhaps the most dramatic progress of the workshop was made by the group working on problem 4. There, a link was made between the appearance of Type II sums and the phenomenon of arithmetic stratification that is the subject of the Manin conjectures. It was felt by most if not all the participants that this went a very long way to providing a clear connection between the circle method and the method of Conrey and Keating. Briefly, the idea is as follows. In the moment calculations for zeta that the latter method performs, the Type II sums must be added to the previously known Type I main terms to get the correct answer. Conrey and Keating determined this essentially by reverse engineering the known answer given by the "recipe" for moments. In other words, it was still not completely clear why these terms emerge. The new insight is that the Type II terms correspond to large contributions to the divisor correlation sums used in calculating moments of long Dirichlet polynomials from counting lattice points on lower dimensional subvarieties. This is a lovely insight that at the same time clears up the connection between the Conrey-Keating method and the circle method.