Crouzeix's conjecture holds for 3×3 matrices with elliptic numerical range centered at an eigenvalue

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OVERVIEW

This presentation is mainly on a paper in progress:

The life of the paper

- Wanted to find a class of matrices with elliptic numerical range to extend 2×2 case, in order to get into research on the conjecture.
- In a paper by Brown & Spitkovsky[1], we found the class

$$\begin{bmatrix} a_1 & b_1 & 0 \\ c_1 & a_2 & b_2 \\ 0 & c_2 & a_1 \end{bmatrix}, \qquad a_k, b_k, c_k \in \mathbb{C}.$$

- We submitted the case $\begin{bmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{bmatrix}$ to SIMAX, but a very helpful referee gave us a good start for the general elliptic 3×3 case.
- We chose to study $\begin{bmatrix} a & b_1 & 0 \\ c_1 & a & b_2 \\ 0 & c_2 & a \end{bmatrix}$; the method imposes $a_1 = a_2$.

[1] E. S. Brown and I. M. Spitkovsky, *On matrices with elliptical numerical ranges*, Linear Multilinear Algebra, 52 (2004), pp. 177–193.

The similarity method

Crouzeix's conjecture: For every square matrix, the numerical range is a 2-spectral set, i.e., for all square matrices A and polynomials p:

$$\|p(A)\| \leq 2 \max_{z \in W(A)} |p(z)|. \tag{1}$$

One way of proving (1): First find a conformal mapping f from the interior of $W(A)$ to $\mathbb{D}$ and extend it continuously to $W(A) \to \overline{\mathbb{D}}$.

Next find an invertible X with $\kappa(X) \leq 2$, such that $\|X\, f(A)\, X^{-1}\| \leq 1$.

Then, by von Neumann's inequality, we get for any polynomial p that

$$\begin{aligned} \|p(A)\| &= \|X^{-1}\left(p \circ f^{-1}(X\, f(A)\, X^{-1})\right) X\| \\ &\leq \|X^{-1}\|\, \|X\|\, \|(p \circ f^{-1})(X\, f(A)\, X^{-1})\| \\ &\leq 2 \max_{w \in \overline{\mathbb{D}}} |(p \circ f^{-1})(w)| = 2 \max_{z \in W(A)} |p(z)|. \end{aligned}$$

This establishes (1) for A.

PARAMETRIZING THE FAMILY A

We follow the approach in the 2×2 case.[2] It turns out that the conjecture holds for all 3×3 matrices with elliptic numerical range centered at an eigenvalue if and only if it holds for the real matrices

$$A := \begin{bmatrix} 1 & q/r & r^2 - 1/r^2 \\ 0 & 0 & qr \\ 0 & 0 & -1 \end{bmatrix}, \qquad q \geq 0, \quad 0 < r \leq 1. \tag{2}$$

Then $\sigma(A) = \{1, 0, -1\}$ and $W(A)$ elliptic with foci ± 1. There is $\rho > 1$ s.t. the major axis of $W(A)$ is $\rho + 1/\rho$ and the minor axis is $\rho - 1/\rho$.

Parametrizing A with (ρ, r) has the major advantage that $W(A)$ is constant for ρ constant. Hence, sometimes set in (2):

$$q^2 = \frac{\left(\rho + 1/\rho\right)^2 - \left(r^2 + 1/r^2\right)^2}{r^2 + 1/r^2}, \qquad \rho > 1, \quad 1/\sqrt{\rho} \leq r \leq 1\,.$$

[2] M. CROUZEIX, *Bounds for analytical functions of matrices*, Integral Equations Operator Theory, 48 (2004), pp. 461–477.

THE CONFORMAL MAPPING

An explicit conformal mapping from the interior of $W(A)$ into $\mathbb{D}$: [3]

$$f(z) = \frac{2z}{\rho} \exp\left(\sum_{n=1}^{\infty} \frac{2\,(-1)^n\, T_{2n}(z)}{n\,(1+\rho^{4n})} \right),$$

where T_{2n} is the $2n$:th Chebyshev polynomial of the first kind.

Thanks to $\sigma(A) = \{-1, 0, 1\}$, we have $f(A) = cA$, where $c := f(1)$.

A lemma (connects to q-Pochhammer symbols): $0 < c(\rho) < 1$ and

$$c(\rho) = \frac{2 \prod_{n=1}^{\infty} (1+\rho^{-8n})^2}{\rho \prod_{n=1}^{\infty} (1+\rho^{4-8n})^2} \le \frac{2\,(1+\rho^{-8})^2\,(1+\rho^{-16})^2}{\rho\cdot(1+\rho^{-4})^2\,(1+\rho^{-12})^2} \le \frac{2}{\rho}.$$

[3] P. HENRICI, *Applied and computational complex analysis. Vol. 3*, Wiley, 1986, pp. 373–374. (Note that a number 2 is missing in the formula for f in the book.)

PARAMETRIZING THE SIMILARITIES X

WLOG X is upper triangular with any diagonal element normalized to 1, but we add the extra assumption that $1 \in \sigma(X^*X)$ in order to simplify characterizing $\kappa(X) \leq 2$. It suffices to consider real X.

A proposition: The matrices

$$X = \begin{bmatrix} s & t & u \\ 0 & 1 & v \\ 0 & 0 & w \end{bmatrix}, \qquad s, t, u, v, w \in \mathbb{R},$$

are invertible with $\kappa(X) = 2$ if: $1/2 \leq sw \leq 2$,

$$2tuv = s^2v^2 - t^2 + t^2v^2 + t^2w^2 - v^2,$$

and

$$\frac{5}{2}sw = s^2 + t^2 + u^2 + v^2 + w^2.$$

Finding X with $\kappa(X) \leq 2$ and $c\|XAX^{-1}\| \leq 1$

A first thing to try, is to find an X with $\kappa(X) \leq 2$ which diagonalizes A.

Such an X of the above form exists and can be explicitly calculated iff

$$r^2 + \frac{1}{r^2} \geq \sqrt{\frac{5^2}{4^2} + 2\left(\rho + 1/\rho\right)^2} - \frac{5}{4}. \qquad (3)$$

This establishes the conjecture for the parameter choices (ρ, r) that satisfy (3), but outside of this area we have to find another X.

Did some optimization by hand guided by the numerical solutions of

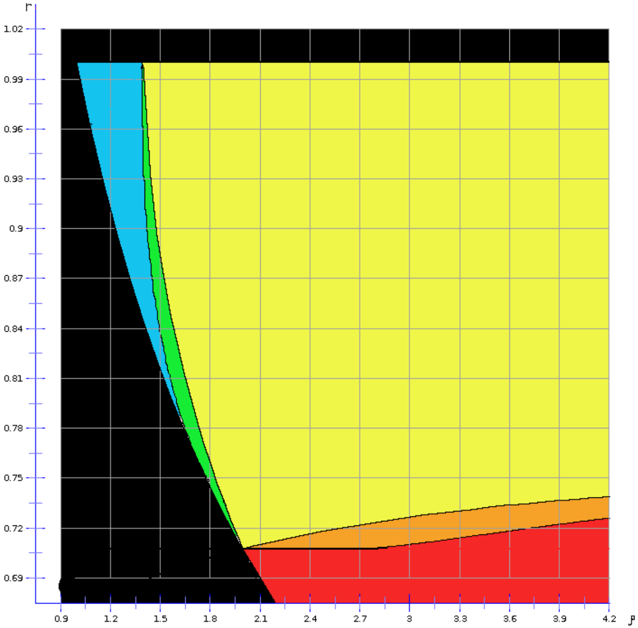
$$\arg\min_X \|X\, f(A)\, X^{-1}\| = \arg\min_X \|XAX^{-1}\| \quad \text{s.t.} \quad \kappa(X) \leq 2.$$

Some trial, error, and good luck gave good enough $X(\rho, r)$.

But then we bumped into another boundary for X and we had to search for a new family of similarity transformations...

THE PATCHWORK

In the end, the picture looks like this:



There is also a cut narrowly covered by the semi-strip $\rho \geq 10$, $0.75 \leq r \leq 0.77$.

On most of these patches, prove that (for $\widetilde{c} \geq c$):

$$\widetilde{c}\|XAX^{-1}\| \leq 1.$$

One gets a high-degree polynomial inequality. The method tailored to patch.

The overlap of the patches almost done.

$\rightarrow$ resubmit

Some questions for the workshop

Similarity transform approach is sharp in completely bounded case.

1. Is it interesting and possible to develop a constructive approach to finding similarity transformations X for which one can verify that $\kappa(X) \leq 2$ and $\|X\, f(A)\, X^{-1}\| \leq 1$? Our way of optimizing gives very little analytical insight.

2. Can the similarity analysis be unified in order to avoid patchwork? For instance, can we find a single family of X?

3. What happens in the matrix as the parameters cross a patch boundary, sometimes even with unchanged numerical range?

4. Could our class of matrices be an interesting example while studying the other, more general approaches suggested?
   Can properties of $W(A)$ be exploited more in the general case?

SOME QUESTIONS FOR THE WORKSHOP

Several participants have expressed interest in Blaschke products g.

5. Let f be a conformal map from the interior of $W(A)$ to $\mathbb{D}$; then set $B := f(A)$. How to compute extremal Blaschke products:

$$\psi_{\mathbb{D}}(B) := \max_{\alpha_j} \{\|g(B)\|;\ g(z) = \prod_{j=1}^{m} \frac{z-\alpha_j}{1-\overline{\alpha}_j z},\ \alpha_j \in \mathbb{D},\ m \le n-1\}\,?$$

6. How to apply Henrici's conformal map

$$f(z) = \frac{2z}{\rho} \exp\left(\sum_{n=1}^{\infty} \frac{2\,(-1)^n\, T_{2n}(z)}{n\,(1+\rho^{4n})} \right)$$

in cases with elliptic numerical range and $\sigma(A) = \{\sigma_0, \pm 1\}$, where $\sigma_0 \neq 0$? In this case $f(A) \neq f(1)A$?