Some Extensions of the Crouzeix-Palencia Result

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Crouzeix's Conjecture, Palencia's Result

Crouzeix's Conjecture (2004): For any n by n matrix A (or any bounded linear operator on a Hilbert space) and any polynomial p (or any function analytic in $W(A)$),

$$\|p(A)\| \leq 2\|p\|_{W(A)},$$

where $\| \cdot \|$ denotes the operator 2-norm (largest singular value) and $\| \cdot \|_{W(A)}$ denotes the ∞-norm on the numerical range: $W(A) := \{\langle Aq, q\rangle : \langle q, q\rangle = 1\}$. In other words, $W(A)$ is a 2-spectral set for A.

Crouzeix proved 11.08 instead of 2.

Crouzeix's Conjecture, Palencia's Result, cont.

Palencia's Result (announced summer of 2016):

$$\|p(A)\| \leq (1+\sqrt{2})\|p\|_{W(A)},$$

Proof (in form due to Crouzeix and Palencia) can be found in The Numerical Range is a $(1+\sqrt{2})$-Spectral Set, *SIAM J. Matrix Anal. Appl., 38 (2017), pp. 649-655.*

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I will discuss some ways that their arguments can be extended to find other K-spectral sets for some moderate values of K. (For simplicity, will assume here that A is a square matrix.)

Crouzeix-Palencia Proof of $1 + \sqrt{2}$

Cauchy integral formula: For $f \in \mathcal{A}(\Omega)$,

$$\begin{aligned} f(A) &= \frac{1}{2\pi i} \int_{\partial\Omega} (\sigma I - A)^{-1} f(\sigma)\, d\sigma \\ &= \int_0^L \left[\frac{\sigma'(s)}{2\pi i} R(\sigma(s), A) \right] f(\sigma(s))\, ds. \end{aligned}$$

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New Idea: Look at $g(A)^*$, where

$$g(A) = \int_0^L \left[\frac{\sigma'(s)}{2\pi i} R(\sigma(s), A) \right] \overline{f(\sigma(s))}\, ds.$$

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Note that since $\bar{f} \notin \mathcal{A}(\Omega)$, cannot apply the Cauchy integral formula as above.

Crouzeix-Palencia Proof of $1+\sqrt{2}$, cont.

Lemma 1. The Hermitian matrix

$$\mu(s) := \frac{\sigma'(s)}{2\pi i} R(\sigma(s), A) + \left[\frac{\sigma'(s)}{2\pi i} R(\sigma(s), A) \right]^*$$

is positive semidefinite for all $s \in [0, L]$ if Ω is convex and $W(A) \subset \Omega$.

On tangent line to $W(A)$, the smallest eigenvalue of $\mu(s)$ is 0. On the side containing $W(A)$ it is negative, and on side not containing $W(A)$ it is positive.

Crouzeix-Palencia Proof of $1 + \sqrt{2}$, cont.

Lemma 2. Assume that Ω is convex and $W(A) \subset \Omega$. For $f \in \mathcal{A}(\Omega)$ with $\|f\|_\Omega = 1$, let $S = f(A) + g(A)^*$. Then $\|S\| \leq 2$.

Proof relies heavily on the fact that $\mu(s)$ is positive semidefinite.

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Proof relies heavily on the fact that $\mu(s)$ is positive semidefinite.

Note: In all numerical experiments that I have tried, where A is a square matrix, $\Omega = W(A)$, and f is the function that maximizes $\|f(A)\|/\|f\|_{W(A)}$, $\|f(A)\| \leq \|f(A) + g(A)^*\|$. If we prove this, we are done!

Extension

If Ω does not contain all of $W(A)$, define

$$M(s) := \mu(s) - \lambda_{\min}(\mu(s))\, I.$$

By definition, $M(s)$ is positive semidefinite on $\partial\Omega$.

Extension, cont.

Lemma 2e: Let Ω be a region with smooth boundary containing the spectrum of A in its interior. For $f \in \mathcal{A}(\Omega)$ with $\|f\|_\Omega = 1$, let

$$S = f(A) + g(A)^* + \gamma I, \quad \gamma := -\int_0^L \lambda_{min}(\mu(s)) f(\sigma(s))\, ds.$$

Then $\|S\| \leq 2 + \delta$, where

$$\delta = -\int_0^L \lambda_{min}(\mu(s))\, ds.$$

Proof (almost identical to that of Lemma 2 in CP2017):

Extension, cont.

For any two unit vectors u and v:

$$\begin{aligned}
|u^* S v| &= \left| \int_0^L (u^* M(s) v) f(\sigma(s))\, ds \right| \leq \int_0^L |u^* M(s) v|\, ds \\
&\leq \int_0^L |u^* M(s) u|^{1/2} \cdot |v^* M(s) v|^{1/2}\, ds \text{ (C-S, since } M(s) \text{ PSD)} \\
&\leq \left(\int_0^L u^* M(s) u\, ds \right)^{1/2} \left(\int_0^L v^* M(s) v\, ds \right)^{1/2} \text{ (Hölder's ineq)} \\
&= \left(u^* \left(\int_0^L M(s)\, ds \right) u \right)^{1/2} \left(v^* \left(\int_0^L M(s)\, ds \right) v \right)^{1/2} \\
&= (u^*(2I + \delta) u)^{1/2} (v^*(2I + \delta) v)^{1/2} = 2 + \delta. \quad \square
\end{aligned}$$

Remarks

- δ in Lemma 2e ($\|S\| \leq 2 + \delta$) can be positive or negative. It is 0 if $\Omega = W(A)$, positive if Ω is a proper subset of $W(A)$, and negative if Ω is convex and $W(A)$ is a proper subset of Ω.

Remarks

- δ in Lemma 2e ($\|S\| \leq 2 + \delta$) can be positive or negative. It is 0 if $\Omega = W(A)$, positive if Ω is a proper subset of $W(A)$, and negative if Ω is convex and $W(A)$ is a proper subset of Ω.
- Ω in Lemma 2e need not be convex or even connected. (Unfortunately, for the remaining results relating $\|f(A) + g(A)^*\|$ to $\|f(A)\|$, Ω must be convex.)

Crouzeix-Palencia Proof of $1+\sqrt{2}$, cont.

Now must relate $\|f(A)+g(A)^*\|$ to $\|f(A)\|$. For $g(A)$ as defined previously,

$$g(z) = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\overline{f(\sigma)}}{\sigma - z}\, d\sigma.$$

Lemma 3. If $f \in \mathcal{A}(\Omega)$, Ω convex, then $g \in \mathcal{A}(\Omega)$ and $\|g\|_\Omega \le \|f\|_\Omega$.

Theorem 4. Assume that Ω is convex and $\Omega \supset W(A)$ and let c_Ω denote the smallest constant s.t. $\forall f \in \mathcal{A}(\Omega)$, $\|f(A)\| \le c_\Omega \|f\|_\Omega$. Then $c_\Omega \le 1+\sqrt{2}$.

Extension

Theorem 4e. Fix A. Let Ω be a convex domain with smooth boundary containing the spectrum of A in its interior. Then

$$c_\Omega \leq 1 + \hat{\delta} + \sqrt{2 + 2\hat{\delta} + \hat{\delta}^2}\},$$

where $\hat{\delta} = -\int_0^L \min\{0,\ \lambda_{\min}(\mu(s))\}\, ds$.

Proof (almost identical to that of Theorem 4 in CP2017):

Numerical Studies

How does $\lambda_{min}(\mu(s))$ vary as you move inside or outside $\partial W(A)$?

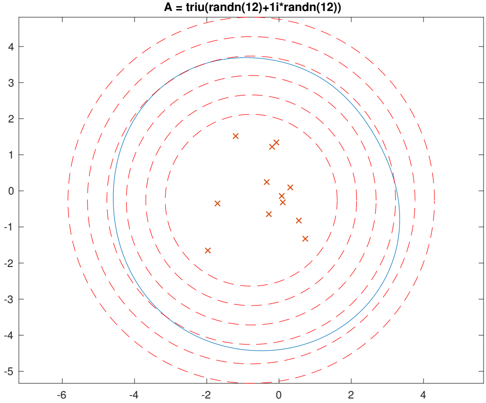

Figure: Eigenvalues (x's) and numerical range (solid curve) of a random complex upper triangular matrix A of dimension $n = 12$. The dashed circles are the ones on which $\lambda_{\min}(\mu(s))$ was computed.

Numerical Studies, cont.

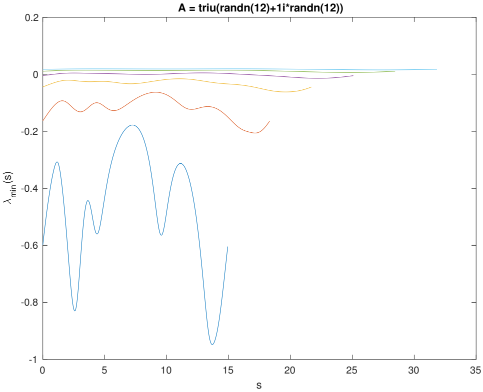

Figure: Plot of $\lambda_{min}(\mu(s))$ vs. arc length s on each of the dashed circles in Figure 1. The bottom curve corresponds to the innermost circle and the curves move up as the circles become larger.

Numerical Studies, cont.

δ	$\hat{\delta}$	$K_{\hat{\delta}}$	K_{disk}	computed c_Ω
7.049	7.049	16.159	9.049	4.400
2.230	2.230	6.610	4.230	2.584
0.721	0.721	3.712	2.721	2.058
0.038	0.078	2.549	2.038	1.752
-0.338	0	2.415	1.662	1.538
-0.576	0	2.415	1.424	1.372

Table: Values of δ, $\hat{\delta}$, upper bound $K_{\hat{\delta}}$ on c_Ω, improved upper bound $K_{disk} = 2 + \delta$, and computed value of c_Ω.

A New Proof of an Old Result

In 1975, Okubo and Ando proved:

If $W(A) \subset \mathbb{D}$, then A is similar to a contraction via a similarity transformation with condition number at most 2: $A = XCX^{-1}$, $\|C\| \leq 1$, $\kappa(X) \leq 2$.
From von Neumann's inequality $\|p(C)\| \leq \sup_{z \in \mathbb{D}} |p(z)|$. Hence $\|p(A)\| \leq 2 \sup_{z \in \mathbb{D}} |p(z)|$; that is, $\mathbb{D}$ is a 2-spectral set for A.

I will show how this can be proved using the Palencia-Crouzeix result.

The Optimal f

When A is an n by n matrix, the form of the function f that satisfies $\|f(A)\| = c_{\Omega}\|f\|_{\Omega}$ is known!

$$f = B \circ \varphi,$$

where φ is any *bijective conformal mapping* from Ω to the unit disk $\mathbb{D}$ and B is a *Blaschke product* of degree at most $n-1$:

$$B(z) = e^{i\phi} \prod_{j=1}^{n-1} \frac{z - \alpha_j}{1 - \bar{\alpha}_j z}, \quad |\alpha_j| \leq 1.$$

Blaschke products map the unit disk to itself.

The Optimal B

Given a matrix Ψ, say, $\Psi = \varphi(A)$, with spectrum in $\mathbb{D}$. What Blaschke product B maximizes $\|B(\Psi)\|$?

Theorem. If $\hat{B}$ maximizes $\|B(\Psi)\|$ and $\|\hat{B}(\Psi)\| > 1$, and if x and y are the left and right singular vectors of $\hat{B}(\Psi)$ corresponding to the largest singular value, then $x \perp y$.

[1] Due to M. Crouzeix, after we observed the result numerically.

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Proof Sketch:[1] Let $M = \hat{B}(\Psi)$ where $\hat{B}$ is optimal. Then no matrix of the form

$$(M - \alpha I)(I - \bar{\alpha} M)^{-1}, \quad |\alpha| < 1$$

can have larger norm than M.

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The Optimal B, cont.

Let y be a unit right singular vector of M corresponding to the largest singular value σ_1 and define $w := (I - \bar{\alpha} M) y$. Then

$$\|(M - \alpha I)(I - \bar{\alpha} M)^{-1} w\| = \|(M - \alpha I) y\| \leq \sigma_1 \|w\| = \sigma_1 \|(I - \bar{\alpha} M) y\|.$$

After some algebra, this becomes

$$2(\sigma_1^2 - 1) \mathrm{Re}(\bar{\alpha} \langle My, y \rangle) \leq |\alpha|^2 (\sigma_1^4 - 1).$$

Since $\sigma_1 > 1$, dividing by $\sigma_1^2 - 1$ and choosing α so that $\bar{\alpha} \langle My, y \rangle = |\alpha| \, |\langle My, y \rangle|$ yields

$$2|\alpha| \, |\langle My, y \rangle| \leq |\alpha|^2 (\sigma_1^2 + 1),$$

and letting $|\alpha| \to 0$, this implies that $|\langle My, y \rangle| = 0$. $\square$

$g(A)^*$ for the Optimal f

Recall that

$$g(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\overline{f(\sigma)}}{\sigma - z}\, d\sigma.$$

If Ω is the unit disk, then it can be shown that for all $z \in \mathbb{D}$, $g(z) \equiv \overline{f(0)}$ so $g(A) = \overline{f(0)}\ I$.

Recall that $\|S\| = \|f(A) + g(A)^*\| \leq 2$. If x and y are normalized left and right singular vectors of $f(A)$ corresponding to the largest singular value, then

$$\|S\| \geq |x^* S y| = |x^* f(A) y + x^* f(0) I y| = \|f(A)\|. \quad \square$$

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If disk does *not* necessarily contain $W(A)$, then $S = f(A) + g(A)^* + \gamma I$, and still have $\|f(A)\| \leq \|S\| \leq 2 + \delta$ (or else $\|f(A)\| \leq 1$).

New Theorem

Theorem. If Ω is any disk containing the spectrum of Ψ in its interior, then Ω is a $\max\{1, 2+\delta\}$-spectral set for Ψ, where

$$\delta = -\int_{\partial\Omega} \lambda_{\min}(\mu(\sigma(s), \Psi)\, ds,$$

$$\mu(\sigma(s), \Psi) = H\left[\frac{\sigma'(s)}{\pi i}(\sigma(s)I - \Psi)^{-1}\right],$$

where $H[\cdot]$ is the Hermitian part.

Corollary

Let φ be a bijective conformal mapping from $W(A)$ to the unit disk $\mathbb{D}$. The unit disk is a $\max\{1, 2 + \delta_{\varphi(A)}\}$-spectral set for $\varphi(A)$, where

$$\delta_{\varphi(A)} = -\int_{\partial\mathbb{D}} \lambda_{\min}(\mu(\sigma(s), \varphi(A))\, ds.$$

Therefore $W(A)$ is a $\max\{1, 2 + \delta_{\varphi(A)}\}$-spectral set for A, since if $K = \max\{1, 2 + \delta_{\varphi(A)}\}$, then

$$\|f(A)\| = \|f \circ \varphi^{-1}(\varphi(A))\| \leq K\|f \circ \varphi^{-1}\|_{\mathbb{D}} = K\|f\|_{W(A)}.$$

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Often $2 + \delta_{\varphi(A)} < 1 + \sqrt{2}$. For example, if $A =$ triu(randn(12)+1i*randn(12)), then $\delta_{\varphi(A)} = 0.0113$, so $W(A)$ is a 2.0113-spectral set for A.

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Let φ be a bijective conformal mapping from $W(A)$ to the unit disk $\mathbb{D}$. The unit disk is a $\max\{1, 2+\delta_{\varphi(A)}\}$-spectral set for $\varphi(A)$, where

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Can we establish *a priori* bounds on the size of $\delta_{\varphi(A)}$?

Summary and Open Problems

- Conjecture will be proved if we can show that: $\|f(A)\| \leq \|f(A) + g(A)^*\|$. Perhaps this can at least be proved for certain classes of matrices. We have proved it for 2 by 2 matrices, some 3 by 3 matrices with elliptical numerical range, matrices whose numerical range is a disk.

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- Proofs rely on the fact that for the optimal B, the left and right singular vectors corresponding to the largest singular value of $B(\varphi(A))$ are orthogonal. Can other properties of the optimal B be established?

Summary and Open Problems, cont.

- Have extended the Crouzeix-Palencia proof to give bounds on c_Ω for regions Ω that do *not* necessarily contain $W(A)$ in terms of the quantities δ, $\hat{\delta}$. Can anything be proved about these quantities? Especially, about

$$\delta_{\varphi(A)} = - \int_{\partial \mathbb{D}} \lambda_{\min}(\mu(\sigma(s), \varphi(A)))\, ds$$

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- What if Ω consists of a union of disjoint regions. Lemma 2e still shows $\|f(A) + g(A)^* + \gamma I\| \leq 2 + \delta$, but how is this related to $\|f(A)\|$? What is the "optimal" f like for, say, a union of two disjoint disks?

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- Paper available at: **https://arxiv.org/pdf/1707.08603.pdf** .

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Now Michael Overton will talk about some extensive numerical experiments that we have performed and associated variational analysis of the Crouzeix ratio...