Mapping theorems for numerical range

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Notation

- $H$ is a complex Hilbert space.
- $T$ is a bounded linear operator on $H$.
- $\sigma(T)$ is the *spectrum* of $T$, namely
  \[ \sigma(T) := \{ \lambda \in \mathbb{C} : (\lambda I - T) \text{ is not invertible} \} . \]
- $W(T)$ is the *numerical range* of $T$, namely
  \[ W(T) := \{ \langle Tx, x \rangle : x \in H, \|x\| = 1 \} . \]
- $w(T)$ is the *numerical radius* of $T$, namely
  \[ w(T) := \sup \{ |\lambda| : \lambda \in W(T) \} . \]
Some basic facts about numerical range

- $W(T)$ is bounded and $\|T\|/2 \leq w(T) \leq \|T\|$.
- $W(T)$ is convex (Toeplitz–Hausdorff theorem).
- $W(T)$ is compact if $\text{dim } H < \infty$.
- $\lambda$ an eigenvalue of $T \Rightarrow \lambda \in W(T)$.
- $\text{conv}(\sigma(T)) \subset \overline{W(T)}$, with equality if $T$ is normal (Berberian)

**Example 1**

If $T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $\sigma(T) = \{0\}$ and $W(T) = \overline{D}$.

**Example 2**

If $T$ is the backward shift on $\ell^2$, then $\sigma(T) = \overline{D}$ and $W(T) = D$. 
Spectral mapping theorem

If \( f \) is holomorphic on a neighborhood of \( \sigma(T) \), then

\[
\sigma(f(T)) = f(\sigma(T)).
\]

Problem

What is the analogue for \( W(T) \)?

- \( W(f(T)) = f(W(T)) \) if \( f(z) = az + b \).
- \( W(f(T)) \subseteq \text{conv}(f(W(T))) \) if \( T \) is normal.
Two examples

Example 1

Let $T := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (normal) and $f(z) := z - z^2$. Then:

- $f(W(T)) = f([0, 1]) = [0, \frac{1}{4}]$,
- $W(f(T)) = \{0\}$ (because $f(T) = T - T^2 = 0$).

Conclusion: $W(f(T)) \not\supset f(W(T))$ in this case.

Example 2

Let $T := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $f(z) := \frac{z + 1/2}{1 + z/2}$. Then:

- $f(W(T)) = f(\mathbb{D}) = \mathbb{D}$,
- $W(f(T)) = \overline{D}(\frac{1}{2}, \frac{3}{4})$ (because $f(T) = \frac{1}{2}I + \frac{3}{4}T$).

Conclusion: $W(f(T)) \not\subset f(W(T))$ in this case.
The power inequality

**Theorem**

\[ w(T^n) \leq w(T)^n \quad (n \geq 1). \]

**Reformulation as a mapping theorem:**

\[ W(T) \subset \overline{D} \implies W(T^n) \subset \overline{D}. \]

- Conjectured by Halmos (early 60’s?)
- Bernau–Smithies (’63): case \( n = 2^k \).
- Lax–Wendroff (’64) \( w(T^n) \leq Kw(T)^n \) where \( K = K(\dim H) \).
- Berger (’65): general result.
- Pearcy (’66): elementary proof.
Theorem (Berger–Stampfli, '67)

Assume that $f(\overline{D}) \subset \overline{D}$ and $f(0) = 0$. Then

$$W(T) \subset \overline{D} \Rightarrow W(f(T)) \subset \overline{D}. \quad (\star)$$

- Generalizes the power inequality.
- Example 2 shows the condition $f(0) = 0$ cannot be dropped. More on this later.
The convex kernel of compact $E \subseteq \mathbb{C}$ is the set of $z \in E$ such that $E$ is star-shaped with respect to $z$. It is a compact convex set.

**Theorem (Kato, ’65)**

Let $f$ be a rational function such that $f(\infty) = \infty$. Let $F$ be a compact convex subset of $\mathbb{C}$, let $E := f^{-1}(F)$, and let $C$ be the convex kernel of $E$. Then

$$W(T) \subseteq C \implies W(f(T)) \subseteq F.$$

- Also generalizes the power inequality.
- Can be partially ‘unified’ with the Berger–Stampfli theorem (Putinar–Sandberg, ’05).
Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Re} \, z > 0\}$.

**Theorem (Kato, ’65)**

If $f(\mathbb{H}) \subset C$, where $C$ is a closed convex set, then

$$W(T) \subset \mathbb{H} \Rightarrow W(f(T)) \subset C.$$ 

If $f(\mathbb{D}) \subset \mathbb{H}$, then

$$W(T) \subset \mathbb{D} \Rightarrow W(f(T)) \subset \mathbb{H} - \text{Re} \, f(0).$$

**Example:**

Let $T := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $f(z) := t \left( \frac{1+z}{1-z} \right)$, where $t > 0$.

Then $f(\mathbb{D}) = \mathbb{H}$ and $W(f(T)) = D(t, 2t)$. 
Drury’s theorem

Theorem (Drury, ’08)

Assume that $f(\overline{D}) \subset \overline{D}$. Then

$$W(T) \subset \overline{D} \Rightarrow W(f(T)) \subset \text{conv}\left(\overline{D} \cup \overline{D}(\alpha, 1 - |\alpha|^2)\right),$$

where $\alpha := f(0)$. In particular,

$$W(T) \subset \overline{D} \Rightarrow W(f(T)) \subset (5/4)\overline{D}.$$
Another example

Let $a > b > 0$ and set $c := \sqrt{a^2 - b^2}$. It is well known that, if

$$ T := \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix}, $$

then $\sigma(T) = \{ -c, c \}$, and $W(T)$ is the ellipse

$$ W(T) = \left\{ x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\}. $$

**Theorem (Crouzeix)**

If $f : W(T) \to \overline{\mathbb{D}}$ is a conformal map, then $W(f(T)) \not\subset \overline{\mathbb{D}}$.

- It does not matter which point $f$ sends to 0.
- If $f(0) = 0$, there is a simple proof using Schwarz's lemma.
- Similar phenomenon for a square.
Relation with spectral sets

A compact set $X \subset \mathbb{C}$ is called a $K$-spectral set for $T$ if $\sigma(T) \subset X$ and if, for all functions $f$ holomorphic on a neighborhood of $X$,

$$\|f(T)\| \leq K\|f\|_X.$$ 

**Theorem (Crouzeix)**

If $X$ is a $K$-spectral set for $T$ then, for all $f$ holomorphic near $X$,

$$w(f(T)) \leq \frac{1}{2}(K + K^{-1})\|f\|_X.$$ 

**Applications**

- $W(T) \subset \overline{D} \Rightarrow \overline{D}$ is a 2-spectral set for $T$ (Okubo–Ando, '75). Thus if $W(T) \subset \overline{D}$ and $f(\overline{D}) \subset \overline{D}$, then $W(f(T)) \subset (5/4)\overline{D}$.

- $W(T)$ is $(1 + \sqrt{2})$-spectral set for $T$ (Crouzeix–Palencia, '17). Hence, if $f(W(T)) \subset \overline{D}$, then $W(f(T)) \subset \sqrt{2}\overline{D}$. 

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A compact $X \subset \mathbb{C}$ is a complete $K$-spectral set for $T$ if $\sigma(T) \subset X$ and if, for all matrix-valued functions $F$ holomorphic near $X$,

$$\|F(T)\| \leq K \|F\|_X.$$ 

**Theorem (Arveson, ’72)**

$X$ is complete $1$-spectral set for $T$ iff $T$ has a normal $\partial X$-dilation.

**Theorem (Paulsen, ’84)**

$X$ is a complete $K$-spectral set for $T$ iff there exists $S \in B(H)$ such that $\|S\|\|S^{-1}\| \leq K$ and $X$ is a complete $1$-spectral set for $STS^{-1}$.

**Theorem (Davidson–Paulsen–Woerdeman ’16)**

The following are equivalent:

- $X$ is a complete $K$-spectral set for $T$.
- $X$ is a complete $\frac{1}{2}(K + K^{-1})$-numerical radius set for $T$. 

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Towards a general mapping theorem?

**Notation**

Given closed convex subsets $C_1$ and $C_2$ of $\mathbb{C}$, let

$$W(C_1, C_2) := \bigcup \left\{ W(f(T)) : W(T) \subset C_1, \ f(C_1) \subset C_2 \right\}.$$ 

Here the union is taken over:

- all operators $T$ (on any $H$) such that $W(T) \subset C_1$, and
- all functions $f$ holomorphic near $C_1$ such that $f(C_1) \subset C_2$.

**Easy facts:**

- $W(C_1, C_2) \supset C_2$.
- $W(C_1, C_2) \subset W(C_1, C_2')$ if $C_2 \subset C_2'$.
- $W(C_1, aC_2 + b) = aW(C_1, C_2,) + b$
Examples

Reminder:

\[
W(C_1, C_2) := \bigcup \left\{ W(f(T)) : W(T) \subset C_1, \ f(C_1) \subset C_2 \right\}.
\]

Examples

- \( W(\overline{D}, \overline{D}) = (5/4)\overline{D} \) (Drury, '08).
- \( W(C_1, \overline{D}) \subset \sqrt{2} \overline{D} \) (Crouzeix–Palencia, '17).
- \( W(\overline{H}, C_2) = C_2 \) (Kato, '65).
- \( W(\overline{D}, \overline{H}) = \mathbb{C} \) (Kato, '65).

Together with the ‘easy facts’, the Crouzeix–Palencia result yields

\[
W(C_1, C_2) \subset \bigcap \left\{ \sqrt{2} \overline{D} : \text{closed disks } \overline{D} \supset C_2 \right\}.
\]
**Possible questions for discussion**

Reminder:

\[ W(C_1, C_2) := \bigcup \left\{ W(f(T)) : W(T) \subset C_1, f(C_1) \subset C_2 \right\}. \]

- Is \( W(C_1, C_2) \) the ‘right’ object to consider?
- If so, then can we give a concrete description of it?
- Can we at least show that \( W(C_1, C_2) \) is closed in \( \mathbb{C} \)? Convex?
- What about ‘basepoints’ (as in the Berger–Stampfli theorem)?
- Is there a ‘complete’ version of \( W(C_1, C_2) \)?