Mapping theorems for numerical range

Thomas Ransford

Université Laval, Québec

Workshop on Crouzeix's conjecture
AIM, San Jose, Aug 2017

Notation

- H is a complex Hilbert space.
- T is a bounded linear operator on H.
- $\sigma(T)$ is the *spectrum* of T, namely

$$\sigma(T) := \{\lambda \in \mathbb{C} : (\lambda I - T) \text{ is not invertible}\}.$$

- $W(T)$ is the *numerical range* of T, namely

$$W(T) := \{\langle Tx, x\rangle : x \in H,\ \|x\| = 1\}.$$

- $w(T)$ is the *numerical radius* of T, namely

$$w(T) := \sup\{|\lambda| : \lambda \in W(T)\}.$$

Some basic facts about numerical range

- $W(T)$ is bounded and $\|T\|/2 \le w(T) \le \|T\|$.
- $W(T)$ is convex (Toeplitz–Hausdorff theorem).
- $W(T)$ is compact if $\dim H < \infty$.
- λ an eigenvalue of $T \Rightarrow \lambda \in W(T)$.
- $\operatorname{conv}(\sigma(T)) \subset \overline{W(T)}$, with equality if T is normal (Berberian)

Example 1

If $T = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, then $\sigma(T) = \{0\}$ and $W(T) = \overline{\mathbb{D}}$.

Example 2

If T is the backward shift on ℓ^2, then $\sigma(T) = \overline{\mathbb{D}}$ and $W(T) = \mathbb{D}$.

Mapping theorems

Spectral mapping theorem

If f is holomorphic on a neighborhood of $\sigma(T)$, then

$$\sigma(f(T)) = f(\sigma(T)).$$

Problem

What is the analogue for $W(T)$?

- $W(f(T)) = f(W(T))$ if $f(z) = az + b$.
- $W(f(T)) \subset \overline{\text{conv}}(f(W(T)))$ if T is normal.

Two examples

Example 1

Let $T := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (normal) and $f(z) := z - z^2$. Then:

- $f(W(T)) = f([0,1]) = [0, \frac{1}{4}]$,
- $W(f(T)) = \{0\}$ (because $f(T) = T - T^2 = 0$).

Conclusion: $W(f(T)) \not\supset f(W(T))$ in this case.

Example 2

Let $T := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $f(z) := \dfrac{z + 1/2}{1 + z/2}$. Then:

- $f(W(T)) = f(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$,
- $W(f(T)) = \overline{D}(\frac{1}{2}, \frac{3}{4})$ (because $f(T) = \frac{1}{2}I + \frac{3}{4}T$).

Conclusion: $W(f(T)) \not\subset f(W(T))$ in this case.

The power inequality

Theorem

$$w(T^n) \le w(T)^n \quad (n \ge 1).$$

Reformulation as a mapping theorem:

$$W(T) \subset \overline{\mathbb{D}} \quad \Rightarrow \quad W(T^n) \subset \overline{\mathbb{D}}.$$

- Conjectured by Halmos (early 60's?)
- Bernau–Smithies ('63): case $n = 2^k$.
- Lax–Wendroff ('64) $w(T^n) \le K w(T)^n$ where $K = K(\dim H)$.
- Berger ('65): general result.
- Pearcy ('66): elementary proof.

Berger–Stampfli theorem

Theorem (Berger–Stampfli, '67)

Assume that $f(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$ *and* $f(0) = 0$. *Then*

$$W(T) \subset \overline{\mathbb{D}} \quad \Rightarrow \quad W(f(T)) \subset \overline{\mathbb{D}}.$$

- Generalizes the power inequality.
- Elementary proof using Blaschke products (Klaja–Mashreghi–R, 2016).
- Example 2 shows the condition $f(0) = 0$ cannot be dropped. More on this later.

Kato's theorems

The *convex kernel* of compact $E \subset \mathbb{C}$ is the set of $z \in E$ such that E is star-shaped with respect to z. It is a compact convex set.

Theorem (Kato, '65)

Let f be a rational function such that $f(\infty) = \infty$. Let F be a compact convex subset of $\mathbb{C}$, let $E := f^{-1}(F)$, and let C be the convex kernel of E. Then

$$W(T) \subset C \quad \Rightarrow \quad W(f(T)) \subset F.$$

- Also generalizes the power inequality.
- Can be partially 'unified' with the Berger–Stampfli theorem (Putinar–Sandberg, '05).

Kato's theorems (continued)

Let $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Theorem (Kato, '65)

If $f(\overline{\mathbb{H}}) \subset C$, where C is a closed convex set, then

$$W(T) \subset \overline{\mathbb{H}} \quad \Rightarrow \quad W(f(T)) \subset C.$$

If $f(\overline{\mathbb{D}}) \subset \overline{\mathbb{H}}$, then

$$W(T) \subset \overline{\mathbb{D}} \quad \Rightarrow \quad W(f(T)) \subset \overline{\mathbb{H}} - \operatorname{Re} f(0).$$

Example:

Let $T := \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $f(z) := t\Bigl(\dfrac{1+z}{1-z}\Bigr)$, where $t > 0$.

Then $f(\overline{\mathbb{D}}) = \overline{\mathbb{H}}$ and $W(f(T)) = \overline{D}(t, 2t)$.

Drury's theorem

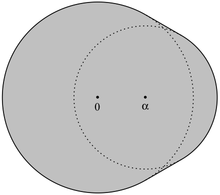
Theorem (Drury, '08)

Assume that $f(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$. *Then*

$$W(T) \subset \overline{\mathbb{D}} \;\Rightarrow\; W(f(T)) \subset \operatorname{conv}\Big(\overline{\mathbb{D}} \cup \overline{D}(\alpha, 1 - |\alpha|^2)\Big),$$

where $\alpha := f(0)$. *In particular,*

$$W(T) \subset \overline{\mathbb{D}} \quad \Rightarrow \quad W(f(T)) \subset (5/4)\overline{\mathbb{D}}.$$



Another example

Let $a > b > 0$ and set $c := \sqrt{a^2 - b^2}$. It is well known that, if

$$T := \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix},$$

then $\sigma(T) = \{-c, c\}$, and $W(T)$ is the ellipse

$$W(T) = \left\{ x + iy : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \right\}.$$

Theorem (Crouzeix)

If $f : W(T) \to \overline{\mathbb{D}}$ is a conformal map, then $W(f(T)) \not\subset \overline{\mathbb{D}}$.

- It does not matter which point f sends to 0.
- If $f(0) = 0$, there is a simple proof using Schwarz's lemma.
- Similar phenomenon for a square.

Relation with spectral sets

A compact set $X \subset \mathbb{C}$ is called a *K-spectral set* for T if $\sigma(T) \subset X$ and if, for all functions f holomorphic on a neighborhood of X,

$$\|f(T)\| \le K\|f\|_X.$$

Theorem (Crouzeix)

If X is a K-spectral set for T then, for all f holomorphic near X,

$$w(f(T)) \le \tfrac{1}{2}(K + K^{-1})\|f\|_X.$$

Applications

- $W(T) \subset \overline{\mathbb{D}} \Rightarrow \overline{\mathbb{D}}$ is a 2-spectral set for T (Okubo–Ando, '75). Thus if $W(T) \subset \overline{\mathbb{D}}$ and $f(\overline{\mathbb{D}}) \subset \overline{\mathbb{D}}$, then $W(f(T)) \subset (5/4)\overline{\mathbb{D}}$.
- $W(T)$ is $(1+\sqrt{2})$-spectral set for T (Crouzeix-Palencia, '17). Hence, if $f(W(T)) \subset \overline{\mathbb{D}}$, then $W(f(T)) \subset \sqrt{2}\,\overline{\mathbb{D}}$.

Complete spectral sets

A compact $X \subset \mathbb{C}$ is a *complete K-spectral set* for T if $\sigma(T) \subset X$ and if, for all **matrix-valued** functions F holomorphic near X,

$$\|F(T)\| \leq K\|F\|_X.$$

Theorem (Arveson, '72)

X is complete 1-spectral set for T iff T has a normal ∂X-dilation.

Theorem (Paulsen, '84)

X is a complete K-spectral set for T iff there exists $S \in B(H)$ such that $\|S\|\|S^{-1}\| \leq K$ and X is a complete 1-spectral set for STS^{-1}.

Theorem (Davidson–Paulsen–Woerdeman '16)

The following are equivalent:

- *X is a complete K-spectral set for T.*
- *X is a complete $\frac{1}{2}(K + K^{-1})$-numerical radius set for T.*

Towards a general mapping theorem?

Notation

Given closed convex subsets C_1 and C_2 of $\mathbb{C}$, let

$$W(C_1, C_2) := \bigcup \Big\{ W(f(T)) : W(T) \subset C_1,\ f(C_1) \subset C_2 \Big\}.$$

Here the union is taken over:

- all operators T (on any H) such that $W(T) \subset C_1$, and
- all functions f holomorphic near C_1 such that $f(C_1) \subset C_2$.

Easy facts:

- $W(C_1, C_2) \supset C_2$.
- $W(C_1, C_2) \subset W(C_1, C_2')$ if $C_2 \subset C_2'$.
- $W(C_1, aC_2 + b) = aW(C_1, C_2,) + b$

Examples

Reminder:

$$W(C_1, C_2) := \bigcup \Big\{ W(f(T)) : W(T) \subset C_1,\ f(C_1) \subset C_2 \Big\}.$$

Examples

- $W(\overline{\mathbb{D}}, \overline{\mathbb{D}}) = (5/4)\overline{\mathbb{D}}$ (Drury, '08).
- $W(C_1, \overline{\mathbb{D}}) \subset \sqrt{2}\,\overline{\mathbb{D}}$ (Crouzeix–Palencia, '17).
- $W(\overline{\mathbb{H}}, C_2) = C_2$ (Kato, '65).
- $W(\overline{\mathbb{D}}, \overline{\mathbb{H}}) = \mathbb{C}$ (Kato, '65).

Together with the 'easy facts', the Crouzeix–Palencia result yields

$$W(C_1, C_2) \subset \bigcap \Big\{ \sqrt{2}\,\overline{D} : \text{ closed disks } \overline{D} \supset C_2 \Big\}.$$

Possible questions for discussion

Reminder:

$$W(C_1, C_2) := \bigcup \Big\{ W(f(T)) : W(T) \subset C_1,\ f(C_1) \subset C_2 \Big\}.$$

- Is $W(C_1, C_2)$ the 'right' object to consider?
- If so, then can we give a concrete description of it?
- Can we at least show that $W(C_1, C_2)$ is closed in $\mathbb{C}$? Convex?
- What about 'basepoints' (as in the Berger–Stampfli theorem)?
- Is there a 'complete' version of $W(C_1, C_2)$?