Ritt operators, optimal constants and the inverse generator problem

Felix Schwenninger

University of Hamburg (GER)

https://www.math.uni-hamburg.de/home/schwenninger/hp/

felix.schwenninger[at]uni-hamburg.de

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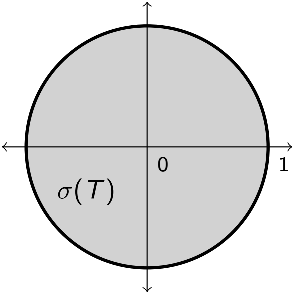
Power-bounded operators

X Banach space, $T : X \to X$ bounded, linear operator

> B is called *power-bounded* if
>
> $$\mathrm{Pb}(T) := \sup_{n \in \mathbb{N}} \|T^n\| < \infty.$$



T power-bounded $\implies \sigma(T) \subset \overline{\mathbb{D}}$,

Resolvent conditions I

Let $T : X \to X$ be such that $\sigma(T) \subset \overline{\mathbb{D}}$ and

$$M_T := \sup_{|\lambda|>1} \|(|\lambda| - 1)R(\lambda, T)\| < \infty. \qquad (*)$$

Note: T power-bounded $\implies$ $M_T \leq Pb(T) < \infty$.

Q: *Does* $(*)$ *imply that* $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$?

Kreiss-Matrix-Theorem '62-'93 [Kreiss,Morton,Strang,Tadmor,LeVeque&Trefethen,Spijker et. al]

Let $T \in \mathbb{C}^{m\times m}$ *such that* $(*)$ *holds. Then*

$$\sup_{n\in\mathbb{N}} \|T^n\| \leq \mathrm{e} \cdot m \cdot M_T.$$

Theorem does not hold for ∞-dimensional Banach spaces

I. Ritt operators (*Tadmor–Ritt* operators)

Resolvent conditions II

$T : X \to X$ bounded with $\sigma(T) \subset \overline{\mathbb{D}}$ and

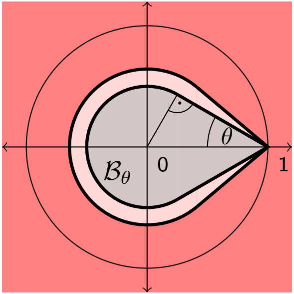
$$C_T = \sup_{|\lambda|>1} \|(\lambda - 1)R(\lambda, T)\| < \infty. \qquad (**)$$

Such T is called a *Tadmor-Ritt* operator.

> **Q:** *Does* $(**)$ *imply that* $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$?

- Ritt condition implies Kreiss condition ($M_T \leq C_T$)
- [Ritt'53]: $\lim_{n\to\infty} \frac{1}{n} T^n = 0$
- [Tadmor'86]: $\|T^n\| = \mathcal{O}(\log n)$
- [O. Nevanlinna '93, Nagy&Zemanek'99, Lyubich'99]:
  T Ritt $\iff$ $\sup_n \|T^n\| < \infty$ and $\sup_n \|n(T^n - T^{n-1})\| < \infty$.
- UPDATE: $\sup_n \|T^n\| < \infty$ already in [Bakaev'88] and [Komatsu'68] (I'm grateful to Y. Tomilov for pointing out the latter)

The spectrum of Tadmor–Ritt operators



$$\mathcal{B}_\theta = \mathrm{conv}\left\{1, \sin\theta\overline{\mathbb{D}}\right\},\ \theta \in [0, \tfrac{\pi}{2}].$$

$$C_T = \sup_{|\lambda|>1} \|(\lambda - 1)R(\lambda, T)\|$$

Lemma ([O.Nevanlinna'93])

Let T be a Tadmor-Ritt operator. Then

- $\sigma(T) \subset \mathcal{B}_\theta$ *with* $\cos\theta = \frac{1}{C_T}$.
- $\sup_{z\in\mathbb{C}\setminus\mathcal{B}_\eta} \|(\lambda - 1)R(\lambda, T)\| \leq (1 - \frac{\cos\eta}{\cos\theta})^{-1} \cdot C_T$

Polynomial calculus for Ritt operators

Definition

An operator $T : X \to X$ is called polynomially bounded if

$$\|p(T)\| \lesssim \|p\|_{\infty,\mathbb{D}} \qquad \forall p \in \mathbb{C}[z]$$

- T polynomially bounded $\implies$ T power-bounded
- Hilbert space: T contractive $\implies$ T polynomially bounded (Halmos): Converse?!: wrong in general (Pisier's example)

Theorem ([Le Merdy'98, de Laubenfels'98])

Let T be a Ritt operator on a Hilbert space. Then

$$T \text{ is polynomially bounded} \iff T \text{ similar to contraction.}$$

Remark: Banach space 'version' [Le Merdy'12, F. Lancien]

Polynomial calculus for Ritt operators

Theorem ([S'16, Vitse'05])

Let T be Tadmor-Ritt, and $p(z) = \sum_{k=m}^{n} a_k z^k$ for $0 \leq m \leq n$. Then

$$\|p(T)\| \lesssim C_T(\log C_T + 1) \log \frac{e(n+1)}{m+1} \|p\|_{\infty,\mathbb{D}}.$$

- improves [Vitse'05]: C_T^5
- Corollary: $\sup_{n\in\mathbb{N}} \|T^n\| \lesssim C_T(\log C_T + 1)$

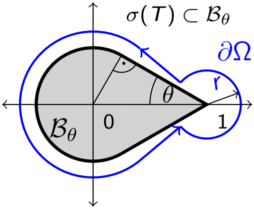
  $=$ Best known power-bound, cf. [Bakaev,Ransford&El-Fallah'02].

  Open problem: Can $C_T(\log C_T + 1)$ be replaced by C_T ??

Proof

T Tadmor-Ritt, $0 \leq m \leq n$, $p(z) = \sum_{k=m}^{n} a_k z^k$.

$$\text{To show} \quad \|p(T)\| \lesssim C_T(\log C_T + 1) \cdot \log \tfrac{e(n+1)}{m+1} \cdot \|p\|_{\infty,\mathbb{D}}.$$



$$p(T) = \frac{1}{2\pi i} \int_{\partial\Omega} p(z) R(z,T)\, dz$$

note: $\Omega \subset (1+r)\mathbb{D}$

Let $r = \frac{1}{n+1}$ and use that

$$\|q\|_{\infty,R\mathbb{D}} \leq R^n \|q\|_{\infty,\mathbb{D}}, \quad R > 1, \deg q = n.$$

The numerical range of Tadmor–Ritt operators

$T : X \to X$ is called an unconditional Ritt operator iff $\exists K > 0$:

$$\|\sum_{n=1}^{m} a_n n(T^n - T^{n-1})\| \leq K \max_{n=1,..,m} |a_n|$$

for any $a_1, .., a_m \in \mathbb{C}$.
Note: unconditional Ritt $\implies$ Ritt [Kalton,Portal'08]

Theorem ([Badea-Seifert'16, F.Lancien-Le Merdy'15,Kalton-Portal'08])

Let $T : X \to X$ and X be a Hilbert space. TFAE

- *T is similar to an operator T_0 such that $W(T_0) \subset \mathcal{B}_\alpha$, $\alpha > 0$*
- *T is an unconditional Ritt operator.*
- *T is similar to contractive Ritt operator*

general Banach spaces: more subtle

II. The continuous case

Sectorial operators of angle $< \frac{\pi}{2}$

An operator $A : D(A) \subset X \to X$ with $\sigma(A) \subset \overline{\mathbb{C}_+}$ and

$$C_A = \sup_{\operatorname{Re}\lambda<0} \|\lambda R(\lambda, A)\| < \infty.$$

is called sectorial operator.

$\implies \exists\, \theta \in [0, \frac{\pi}{2})$: $\sigma(A) \subset S_\theta$, $\sup_{\lambda\in\mathbb{C}\setminus S_\theta} \|\lambda R(\lambda, A)\| < \infty$

Q: *Does* $C_A < \infty$ *imply that* $\sup_{t>0} \|e^{tA}\| < \infty$?

Answer: Yes as A sectorial $\iff$ e^{tA} bounded analytic semigroup

Q: *How does* $\sup_{t>0} \|e^{tA}\|$ *depend on* C_A?

How does $\sup_{t>0} \|e^{tA}\|$ *depend on* C_A*?*

As for Ritt operators, one derives

$$\sup_{t>0} \|e^{tA}\| \lesssim C_A(\log C_A + 1)$$

Q: *Is this* C_A*-dependence optimal?*

III. From continuous to discrete

The Crank-Nicolson scheme (CN)

Let $A \in \mathbb{C}^{N \times N}$

$$\begin{cases} y'(t) & = & Ay(t), & y(0) = y_0 & \text{(diff. eq.)} \\ \frac{y_n - y_{n-1}}{h} & = & A\left(\frac{y_n + y_{n-1}}{2}\right) & n \in \mathbb{N}. & \text{(approx. eq.)} \end{cases}$$

Discretize: $y_n \approx y(n \cdot h)$ with $h > 0$.

$$\Rightarrow \quad y_n = \left(\mathrm{I} + \tfrac{h}{2}A\right)\left(\mathrm{I} - \tfrac{h}{2}A\right)^{-1} y_{n-1},$$

if $\left(\mathrm{I} - \frac{h}{2}A\right)$ is invertible.

$$y_n = \left[\left(\mathrm{I} + \tfrac{h}{2}A\right)\left(\mathrm{I} - \tfrac{h}{2}A\right)^{-1}\right]^n y_0, \qquad n = 1, 2, ..$$

Exact vs. numerical solution

Hence,

$$y(t) = e^{At} y_0, \qquad y_n = T^n y_0,$$

where

$$T = \mathrm{Cay}\left(\tfrac{h}{2}A\right) := \left(\mathrm{I} + \tfrac{h}{2}A\right)\left(\mathrm{I} - \tfrac{h}{2}A\right)^{-1}$$

is called the *Cayley Transform* of $\frac{h}{2}A$.

What is the relation between $y(t)$ and y_n ?

$$\text{If } y \text{ is bounded } \forall y_0 \in X \quad \overset{?}{\Longrightarrow} \quad y_n \text{ bounded } \forall y_0 \in X$$

$$\Updownarrow$$

$$\sup_{t \geq 0} \|e^{tA}\| < \infty \quad \overset{?}{\Longrightarrow} \quad \sup_{n \in \mathbb{N}} \|\mathrm{Cay}(\tfrac{h}{2}A)^n\| < \infty$$

The Cayley Transform Problem: bounded op., ∞-dim.

Let $A \in \mathbb{C}^{N\times N}$, $0 \notin \sigma(A)$ and $\mathrm{Cay}(A) = (\mathrm{I}+A)(\mathrm{I}-A)^{-1}$.

Does $\sup_{t>0} \|e^{tA}\| < \infty$ *imply that* $\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty$ *?*

$$\sup_{t>0} \|e^{tA}\| < \infty \iff \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty.$$

Let $A \in \mathcal{B}(X)$ (bounded op. on Banach space X).

Does $\sup_{t>0} \|e^{tA}\| < \infty$ *imply that* $\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty$ *?*

- e^{tA} still defined via power series.
- Characterisation via spectrum $\sigma(A)$ not valid any more.

If A unbounded $\rightsquigarrow$ Semigroup Theory

Semigroups - abstract exponential functions

Let X be a Banach space.

Definition

A function $T : [0, \infty) \to \mathcal{B}(X)$ is called semigroup of operators if

- $T(0) = \mathrm{I}_X$,
- $T(t + s) = T(t)T(s)$ for all $s, t \geq 0$,
- The mapping $t \mapsto T(t)x$ is continuous for every $x \in X$.

The generator of T is the linear operator A defined by

$$Ax = \lim_{h \to 0^+} \frac{1}{h} \left(T(h)x - x \right)$$

for all $x \in X$ such that the limit exists. A can be unbounded.

Semigroups II

- T is characterized by its generator A

$$\rightsquigarrow T(t) \equiv e^{tA}$$

- For $y_0 \in X$, $y(t) = e^{tA}y_0$ is the solution to

$$y'(t) = Ay(t), \quad y(0) = y_0. \tag{ACP}$$

- If

$$\sup_{t \geq 0} \|e^{tA}\| < \infty,$$

  then e^{tA} is called a bounded semigroup. In this case, the solutions of (ACP) are bounded.

The Cayley Transform Problem: General Form

Let $A : D(A) \subset X \to X$ be such that

- A generates a bounded semigroup e^{tA}, i.e. $\sup_{t\geq 0} \|e^{tA}\| < \infty$.

Question:

Is $\mathrm{Cay}(A)$ power-bounded ??

Notation:

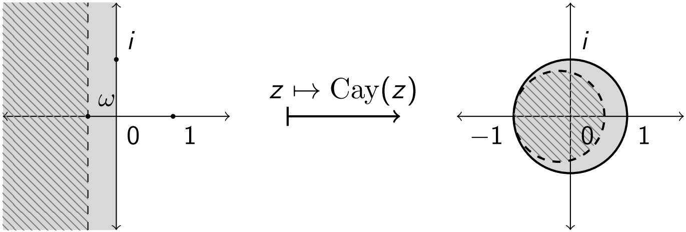
$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty \qquad (CTP_A)$$

$\mathcal{G}_{bdd}$ = generators of bounded semigroups, $\mathrm{Cay}(A) = (\mathrm{I} + A)(\mathrm{I} - A)^{-1}$

CTP: Some Remarks

$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty \qquad (CTP_A)$$

- (??) $\iff$ stability of the Crank–Nicolson scheme.
- $z \mapsto \frac{1+z}{1-z}$ is a *Möbius transformation* such that



Some Answers to the CTP - I

Theorem

1. *There exists an operator A on a Banach space X such that*
   - $A \in \mathcal{G}_{bdd}$
   - $\|\mathrm{Cay}(A)^n\| \approx \sqrt{n}$ *for all* $n \in \mathbb{N}$.
2. *For every* $A \in \mathcal{G}_{bdd}$, $\|\mathrm{Cay}(A)^n\| = \mathcal{O}(\sqrt{n})$.
3. *If* $A \in \mathcal{G}(X)$, *i.e.* e^{tA} *is exponentially stable, then*

$$\|\mathrm{Cay}(A)^n\| = \mathcal{O}(n^{\frac{1}{4}}).$$

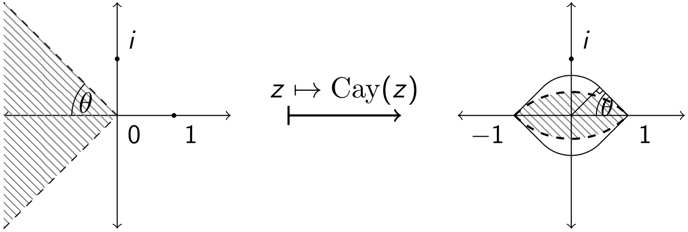
NOTE: An example of a semigroup in 1. is the (left) shift on $L^1(\mathbb{R})$.

$$\Rightarrow \qquad \exists A \in \mathcal{G}_{bdd} : \quad sup_{n\in\mathbb{N}}\|\mathrm{Cay}(A)^n\| = \infty$$

[Brenner-Thomee'79, Crouzeix-Piskarev-Larsson-Thomee'93, Piskarev-Zwart'07, Gomilko, Zwart, Besseling'11]

Some Answers to the CTP - II

$\tau(z) = \frac{1+z}{1-z}$ on sectors:



Theorem ([Palencia'93, Crouzeix-Piskarev-Larsson-Thomee'93])

If A is a sectorial operator, then $\mathrm{Cay}(A)$ is power-bounded.

CTP on Hilbert spaces

In the following let A be defined on a Hilbert space

$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty \qquad (CTP_A)$$

(**??**) is true if

- if e^{tA} is a contraction semigroup (by Lumer–Phillips)
- if $A \in \mathcal{G}_{bdd}$ generates an analytic semigroup [Guo,Zwart '05]
- if $A \in \mathcal{G}_{bdd} \cap \mathcal{B}(X)$ [Gomilko'04,Azizov,Barsukov,Dijksma'04,Guo,Zwart'05]

♠ $A \in \mathcal{G}_{bdd} \implies \sup_n \frac{\|\mathrm{Cay}(A)^n\|}{\log n} < \infty.$ [Gomilko'04,Besseling,Zwart'11]

♣ (**??**) open in general.

The CTP on Hilbert spaces II

In the following we only consider semigroups on Hilbert spaces.

$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty \qquad (CTP_A)$$

Theorem ([S.,Zwart'15])

The following are equivalent.

1. (**??**) *holds for all generators A of bounded semigroups.*
2. (**??**) *holds for all generators A of exp. stable semigroups.*
3. (**??**) *holds for all* $A \in \mathcal{G}_{exp}$ *s.t.* $\exists M \forall t > 0$: $\|e^{tA}\| \leq Me^{-t}$.

NOTE: All assertions hold "for all Hilbert spaces".

Proof of Equivalences of CTP

Consists of two main arguments.

1. Show: *If (CTP_A) holds for all A with $\|e^{tA}\| \leq Me^{\omega t}$ for given M, ω, then $\exists C_{M,\omega} > 0$ s.t. for all these A*

$$\sup_{n \in \mathbb{N}} \|\mathrm{Cay}(A)^n\| < C_{M,\omega}.$$

2. Use the relation to the *Inverse Generator Problem*.

IV. The Inverse Generator Problem

The Inverse Generator Problem: Finite dim.

Let $A \in \mathbb{C}^{N \times N}$ and A invertible.

Does $\sup_{t>0} \|e^{tA}\| < \infty$ *imply that* $\sup_{t>0} \|e^{tA^{-1}}\| < \infty$ *?*

$$\sup_{t>0} \|e^{tA}\| < \infty \iff \sup_{n\in\mathbb{N}} \|e^{tA^{-1}}\| < \infty.$$

The Inverse Generator Problem: General Form

Let $A : D(A) \subset X \to X$ be s.t.

- A generates a bounded semigroup e^{tA}, i.e. $\sup_{t \geq 0} \|e^{tA}\| < \infty$.
- $0 \in \sigma_c(A) \cup \rho(A)$, i.e. A^{-1} exists as (densely defined) operator.

Question:

Does A^{-1} generate a bounded semigroup ??

[deLaubenfels'88]

Notation:

$$A \in \mathcal{G}_{bdd} \;\wedge\; 0 \in \sigma_c(A) \cup \rho(A) \quad \overset{?}{\Longrightarrow} \quad A^{-1} \in \mathcal{G}_{bdd}$$

$\mathcal{G}_{bdd}$ = generators of bounded semigroups.

A counterexample on ℓ^p

Theorem ([Gomilko, Tomilov, Zwart '07])

For any $p \in [1, \infty) \setminus \{2\}$, there exists $A \in \mathcal{B}(\ell^p)$ s.t.

- $\sup_{t>0} \|e^{tA}\| < \infty$,
- $0 \in \sigma_c(A)$, *i.e.* A^{-1} *is densely defined,*
- A^{-1} *does not generate a semigroup.*

In fact,

$$A = -\mathrm{I} + S_l,$$

where S_l is the left-shift on ℓ^p.

Similar example on c_0 [Komatsu'66]. See also [Fackler'16].

Some Remarks about the IGP

$$A \in \mathcal{G}_{bdd} \;\wedge\; 0 \in \sigma_c(A) \cup \rho(A) \quad \overset{?}{\Longrightarrow} \quad A^{-1} \in \mathcal{G}_{bdd} \qquad (IGP_A)$$

- The answer is NO in general, even if A bounded,

$$A = -\mathrm{I} + S_l \in \mathcal{B}(\ell^p), \quad p \neq 2.$$

- [de Laubenfels'88]: YES, e.g. if A is *sectorial*
  $\iff$ $t \mapsto e^{tA}$ has bounded analytic extension to a sector.

- Open for Hilbert spaces.

The IGP on Hilbert spaces I

In Hilbert spaces, the are a few more YES to

$$A \in \mathcal{G}_{bdd} \;\wedge\; 0 \in \sigma_c(A) \cup \rho(A) \quad \overset{?}{\Longrightarrow} \quad A^{-1} \in \mathcal{G}_{bdd} \qquad (IGP_A)$$

- If e^{tA} is a contraction semigroup.

  More precisely, by Lumer–Phillips: If $0 \in \sigma_c(A) \cup \rho(A)$, then

  $$(A \in \mathcal{G} \;\wedge\; \sup_{t>0} \|e^{tA}\| \leq 1) \iff (A^{-1} \in \mathcal{G} \;\wedge\; \sup_{t>0} \|e^{tA^{-1}}\| \leq 1)$$

- If $t \mapsto e^{tA}$ can be analytically extended to a sector. [Guo&Zwart'05]

Exponentially stable Semigroups

Recall: A semigroup e^{tA} is called *exponentially stable* if

$$\exists \omega > 0, M \geq 1 \quad \|e^{tA}\| \leq Me^{-\omega t} \quad \forall t \geq 0. \tag{*}$$

$\mathcal{G}_{exp}(X) = \{A \in \mathcal{G}(X) : e^{tA} \text{ is exp. stable}\}$.

Since (**??**) implies $\{z \in \mathbb{C} : \operatorname{Re} z > -\omega\} \subset \rho(A)$,

$$A \in \mathcal{G}_{exp}(X) \quad \Longrightarrow \quad A^{-1} \in \mathcal{B}(X) \text{ and } e^{tA^{-1}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{-n}.$$

Then the IGP is

$$A \in \mathcal{G}_{exp} \quad \overset{?}{\Longrightarrow} \quad \sup_{t>0} \|e^{tA^{-1}}\| < \infty$$

The IGP on Hilbert spaces II

In the following we only consider semigroups on Hilbert spaces.

$$A \in \mathcal{G}_{bdd} \ \wedge \ 0 \in \sigma_c(A) \cup \rho(A) \quad \overset{?}{\Longrightarrow} \quad A^{-1} \in \mathcal{G}_{bdd} \qquad (IGP_A)$$

Theorem ([S.,Zwart'15])

The following are equivalent.

1. (**??**) *holds for all generators A of bounded semigroups.*
2. (**??**) *holds for all generators A of exp. stable semigroups.*
3. (**??**) *holds for all* $A \in \mathcal{G}_{exp}$ *s.t.* $\exists M \forall t > 0$: $\|e^{tA}\| \leq Me^{-t}$.
4. $\forall A: \quad \left(A \in \mathcal{G}_{bdd} \ \wedge \ 0 \in \sigma_c(A) \cup \rho(A) \quad \Longrightarrow \quad A^{-1} \in \mathcal{G}\right)$

NOTE: All assertions hold "for all Hilbert spaces".

The Cayley Transform Problem and relation to IGP

$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \mathrm{Cay}(A) = (\mathrm{I} + A)(\mathrm{I} - A)^{-1} \text{ power-bounded}$$

Lemma (Gomilko's trick)

If $0 \notin \sigma_p(A)$, $(0, \infty) \subset \rho(A)$, *then for all* $\lambda > 0$

$$\lambda(\lambda - A^{-1})^{-1} = \mathrm{Cay}(2\lambda A - \mathrm{I}).$$

Equivalence of problems

Theorem ([S., Zwart'15])

The following assertions are equivalent (Recall: $f(z) = \frac{1+z}{1-z}$)

1. *The Inverse Generator Problem on Hilbert spaces H;*

$$\forall H \; \forall A \in \mathcal{G}(H) \quad \left(\sup_{t \geq 0} \|e^{tA}\| < \infty \implies \sup_{t \geq 0} \|e^{tA^{-1}}\| < \infty \right).$$

2. *Cayley Transform Problem on Hilbert spaces H;*

$$\forall H \; \forall A \in \mathcal{G}(H) \quad \left(\sup_{t \geq 0} \|e^{tA}\| < \infty \implies \sup_{n \in \mathbb{N}} \|f(A)^n\| < \infty \right).$$

3. *The CT/IG Problem on Hilbert spaces for $A \in \mathcal{G}_{exp}$ s.t. $\exists M > 0 : \|e^{tA}\| \leq Me^{-t}$ for all $t > 0$.*

Summary - CT, IVG problems

1. *Cayley-Transform Problem*

$$A \in \mathcal{G}_{bdd} \quad \overset{?}{\Longrightarrow} \quad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty \qquad (CTP_A)$$

2. *Inverse Generator Problem*

$$A \in \mathcal{G}_{bdd} \ \wedge \ 0 \in \sigma_c(A) \cup \rho(A) \quad \overset{?}{\Longrightarrow} \quad A^{-1} \in \mathcal{G}_{bdd}$$

- No, in general. But open on Hilbert spaces.
- both problems are equivalent.
- suffices to consider exponentially stable semigroups.

Some references (see the references therein)

- FS, *On Functional Calculus Estimates*,
  PhD thesis, University of Twente, The Netherlands, 2015
- FS, *Functional calculus estimates for Tadmor–Ritt operators*
  JMAA, 2016.