On admissible eigenvalue approximations from Krylov subspace methods for non-normal matrices

Jurjen Duintjer Tebbens

joint work with

Gérard Meurant

Crouzeix’s conjecture workshop
Outline

1. Introduction
2. Ritz values
3. Harmonic Ritz values, GMRES and FOM polynomials
1 Introduction
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for a given a nonsingular matrix and nonzero vector

$$A \in \mathbb{C}^{n \times n}, \quad v \in \mathbb{C}^n \quad (\text{w.l.o.g.} \quad ||v|| = 1).$$
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Note that

- all vectors $w \in \mathcal{K}_k(A, v) = \text{span}\{v, Av, \ldots, A^{k-1}v\}$ are of the form
  $$w = \alpha_0 v + \alpha_1 Av + \cdots + \alpha_{k-1} A^{k-1}v = p_{k-1}(A)v$$

  for a polynomial $p_{k-1}$ of degree $k - 1$. 

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We will restrict ourselves to non-normal matrices (for which Crouzeix's conjecture has not yet been proved).
With non-normal matrices, we can very roughly divide the available methods into those using short recurrences (lower computational costs) and those using long recurrences (enhanced stability properties).
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In the $k$th iteration of the process (without breakdown) it computes the decomposition

$$AV_k = V_{k+1} \tilde{H}_k,$$

where the columns of $V_k = [v_1, \ldots, v_k]$ (the Arnoldi vectors) contain an orthogonal basis for the $k$th Krylov subspace $\mathcal{K}_k(A, v)$ and $\tilde{H}_k$ is rectangular upper Hessenberg.
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By deleting the last row we get the square matrix

\[ H_k = V_k^* A V_k \in \mathbb{C}^{k \times k}; \]

\(H_k\) is the orthogonal restriction of \(A\) onto \(\mathcal{K}_k(A, v)\) and \(A\) is a dilation of \(H_k\).
Essentially,

- for eigenpair approximations of $A$, the Arnoldi method [Arnoldi - 1951], [Saad - 1980] uses the eigenvalues and eigenvectors of $H_k$

and the first $k$ Arnoldi vectors. The eigenvalue approximations are called Ritz values, the eigenvector approximations Ritz vectors.
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- Both the GMRES and the Arnoldi method are very popular methods that are successful for a large variety of problem classes.

- Nevertheless, convergence behavior of the two methods is not fully understood, analysis is particularly challenging with highly non-normal input matrices.
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A complete description of the class of matrices and right hand sides with prescribed convergence and eigenvalues was given in [Arioli, Pták, Strakoš - 1998].
Theorem [Greenbaum, Pták & Strakoš, 1996]. Let

\[ 1 = \|b\|_2 = f_0 \geq f_1 \geq f_2 \cdots \geq f_{n-1} > 0 \]

be any non-increasing sequence of real positive values and let

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be any set of nonzero complex numbers. Then there exists a class of matrices \( A \in \mathbb{C}^{n \times n} \) and right-hand sides \( b \in \mathbb{C}^n \) such that the residual vectors \( r_k \) generated by the GMRES method applied to \( A \) and \( b \) satisfy

\[ \|r_k\|_2 = f_k, \quad 0 \leq k \leq n, \quad \text{and} \quad \text{eig}(A) = \{\lambda_1, \ldots, \lambda_n\}. \]
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For other popular methods like Bi-CG and QMR, it was proved as well that convergence behavior can be arbitrarily poor, independent from the eigenvalue distribution [D.T. & Meurant, 2016].
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Although in practice eigenvalues do often influence convergence of GMRES, they cannot be used as a universal tool for explaining GMRES and such a tool is unlikely to exist.
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Consider a tridiagonal Jacobi matrix $T_m$ and its leading principal submatrix $T_k$ for some $k < m$. If the ordered eigenvalues of $T_k$ are

$$\rho_1^{(k)} < \rho_2^{(k)} < \ldots < \rho_k^{(k)},$$

then in every open interval between two subsequent eigenvalues

$$\left( \rho_{i-1}^{(k)}, \rho_i^{(k)} \right), \quad i = 2, \ldots, k,$$

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This interlacing property enables, among others, to prove the persistence theorem (see [Paige - 1971, 1976, 1980] or [Meurant, Strakoš - 2006]) which is crucial for controlling the convergence of Ritz values in the Lanczos method.
There are generalizations of the interlacing property to the non-hermitian but normal case [Fan, Pall - 1957], [Thompson - 1966], [Ericsson - 1990], [Malamud - 2005], [Carden - 2013].
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By arbitrary behavior we mean arbitrary Ritz values for all iterations (we do not consider eigenvectors). Note that this involves many more conditions than prescribing one residual norm per GMRES iteration.
2. Prescribed convergence for Arnoldi’s method

Notation: Let the $k$th Hessenberg matrix $H_k$ generated in Arnoldi’s method have the eigenvalue $\rho$ and eigenvector $y$,

$$H_k y = \rho y.$$ 

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With the Arnoldi decomposition $AV_k = V_{k+1} \tilde{H}_k$, we obtain for the Ritz-value Ritz-vector pair $\{\rho, V_k y\}$ the residual norm:

$$\|A(V_k y) - \rho(V_k y)\| = \|A(V_k y) - V_k H_k y\| = \|V_{k+1} \tilde{H}_k y - V_k H_k y\| = h_{k+1,k} |e_k^T y|.$$
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\]

Often for small \( h_{k+1,k} |e_k^T y| \), the Arnoldi method takes \( \{\rho, V_k y\} \) as an approximate eigenvalue-eigenvector pair of \( A \). Note that a small value \( h_{k+1,k} |e_k^T y| \) needs not imply that \( \rho \) is close to a true eigenvalue of \( A \), see e.g. [Chatelin - 1993], [Godet-Thobie - 1993]; convergence analysis cannot be based on this value but focusses instead on the quality of approximate invariant subspaces [Beattie, Embree, Sorensen - 2005].
Theorem 1 [DT, Meurant - 2012]. Let the set

\[ \mathcal{R} = \{ \rho_1^{(1)}, \]
\[
(\rho_1^{(2)}, \rho_2^{(2)}), \\
\vdots \\
(\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), \\
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represent any choice of \( n(n+1)/2 \) complex Ritz values.
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**Theorem 1** [DT, Meurant - 2012]. Let the set

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represent any choice of \( n(n+1)/2 \) complex Ritz values and denote by \( C^{(k)} \) the companion matrix of the polynomial with roots \( \rho^{(k)}_1, \ldots, \rho^{(k)}_k \), i.e.

\[
C^{(k)} = \begin{pmatrix}
0 & \ldots & 0 & -\alpha_0 \\
1 & 0 & \ldots & 0 & -\alpha_1 \\
& \ddots & \ddots & \ddots & \ddots \\
& & 1 & -\alpha_{k-1} \\
\end{pmatrix}, \quad \prod_{j=1}^{k} (z - \rho^{(k)}_j) = z^k + \sum_{j=0}^{k-1} \alpha_j z^j.
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If we define the unit upper triangular matrix $U(S)$ through

$$U(S) = I_n - \begin{bmatrix} 0 & C^{(1)}e_1 & & \\ 0 & 0 & C^{(2)}e_2 & \\ & & \ddots & \ddots \\ & & & 0 & C^{(n-1)}e_{n-1} \end{bmatrix},$$
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then the upper Hessenberg matrix

$$H(R) = U(S)^{-1} C^{(n)} U(S)$$

has the spectrum $\lambda_1, \ldots, \lambda_n$ and its $k$th leading principal submatrix has spectrum

$$\rho^{(k)}_1, \ldots, \rho^{(k)}_k, \quad k = 1, \ldots, n - 1.$$

It has unit subdiagonal.
2. Prescribed convergence for Arnoldi's method

**Proof:** The $k \times k$ leading principal submatrix of $H(\mathcal{R})$ is

$$[I_k, 0] H(\mathcal{R}) \begin{bmatrix} I_k \\ 0 \end{bmatrix} = [I_k, 0] U(S)^{-1} C^{(n)} U(S) \begin{bmatrix} I_k \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} U_1^{-1}, \tilde{u}_{k+1}, \ldots, \tilde{u}_n \end{bmatrix} \begin{bmatrix} 0 \\ U_k \\ 0 \end{bmatrix} = \begin{bmatrix} U_1^{-1}, \tilde{u}_{k+1} \end{bmatrix} \begin{bmatrix} 0 \\ U_k \end{bmatrix},$$

where $U_k$ denotes the $k \times k$ leading principal submatrix of $U(S)$ and $\tilde{u}_j$ denotes the vector of the first $k$ entries of the $j$th column of $U(S)^{-1}$ for $j > k$. 
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where $U_k$ denotes the $k \times k$ leading principal submatrix of $U(S)$ and $\tilde{u}_j$ denotes the vector of the first $k$ entries of the $j$th column of $U(S)^{-1}$ for $j > k$.

Its spectrum is also the spectrum of the matrix

$$
U_k [U_k^{-1}, \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ U_k \end{bmatrix} U_k^{-1} = [I_k, U_k \tilde{u}_{k+1}] \begin{bmatrix} 0 \\ I_k \end{bmatrix},
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which is a companion matrix with last column $U_k \tilde{u}_{k+1}$. 
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From

\[ e_{k+1} = U_{k+1}U_{k+1}^{-1} e_{k+1} = \begin{bmatrix} U_k & -C^{(k)} e_k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_{k+1} \\ 1 \end{bmatrix} = \begin{bmatrix} U_k \tilde{u}_{k+1} - C^{(k)} e_k \\ 1 \end{bmatrix} \]

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we obtain \( U_k \tilde{u}_{k+1} = C^{(k)} e_k \). \( \square \)

Remark: The matrix

\[ H(\mathcal{R}) = U(S)^{-1} C^{(n)} U(S). \]

is the unique upper Hessenberg matrix \( H(\mathcal{R}) \) with the prescribed spectrum and Ritz values and the entry one along the subdiagonal (see also [Parlett, Strang - 2008] where \( H(\mathcal{R}) \) is constructed in a different way).
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Note that \( U(\mathcal{S}) \) transforms the matrix \( C^{(n)} \) with all Ritz values zero to the matrix \( H(\mathcal{R}) \) with prescribed Ritz values. It is composed of (columns of) companion matrices and we will call \( U(\mathcal{S}) \) the Ritz value companion transform.
Thus the Ritz values generated in the Arnoldi method can exhibit any convergence behavior: It suffices to apply the Arnoldi process with the initial Arnoldi vector $e_1$ and the matrix $H(\mathcal{R})$ with arbitrarily prescribed Ritz values. Then the method generates the Hessenberg matrix $H(\mathcal{R})$ itself.
Thus the Ritz values generated in the Arnoldi method can exhibit any convergence behavior: It suffices to apply the Arnoldi process with the initial Arnoldi vector $e_1$ and the matrix $H(\mathcal{R})$ with arbitrarily prescribed Ritz values. Then the method generates the Hessenberg matrix $H(\mathcal{R})$ itself.

Question: Can the same prescribed Ritz values be generated with positive entries other than one on the subdiagonal?
2. Prescribed convergence for Arnoldi’s method

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For $\sigma_1, \sigma_2, \ldots, \sigma_{n-1} > 0$ consider the diagonal similarity transformation

$$H \equiv \text{diag} \left(1, \sigma_1, \sigma_1 \sigma_2, \ldots, \Pi_{j=1}^{n-1} \sigma_j\right) H(\mathcal{R}) \left(\text{diag} \left(1, \sigma_1, \sigma_1 \sigma_2, \ldots, \Pi_{j=1}^{n-1} \sigma_j\right)\right)^{-1}. $$
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Then the subdiagonal of $H$ has the entries $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and all leading principal submatrices of $H$ are similar the corresponding leading principal submatrices of $H(\mathcal{R})$. 

2. Prescribed convergence for Arnoldi’s method
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**Theorem 2** [DT, Meurant - 2012]. Assume we are given a set of tuples

$$\mathcal{R} = \{ \rho_1^{(1)}, (\rho_1^{(2)}, \rho_2^{(2)}), \ldots, (\rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}), (\lambda_1, \ldots, \lambda_n) \},$$

of complex numbers and $n - 1$ positive real numbers $\sigma_1, \ldots, \sigma_{n-1}$. 
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If \( A \) is a matrix of order \( n \) and \( v \) a unit nonzero \( n \)-dimensional vector, then the following assertions are equivalent:
2. Prescribed convergence for Arnoldi’s method

1. The Hessenberg matrix generated by the Arnoldi method applied to $A$ and initial Arnoldi vector $v$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, subdiagonal entries $\sigma_1, \ldots, \sigma_{n-1}$ and $\rho^{(k)}_1, \ldots, \rho^{(k)}_k$ are the eigenvalues of its $k$th leading principal submatrix for all $k = 1, \ldots, n - 1$. 
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2. The matrix $A$ and initial vector $v$ are of the form

$$A = V D_{\sigma} U(S)^{-1} C^{(n)} U(S) D^{-1}_{\sigma} V^*, \quad v = V e_1,$$
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where $V$ is unitary, $U(S)$ is the Ritz value companion transform,

$$D_\sigma = \text{diag}(1, \sigma_1, \sigma_1 \sigma_2, \ldots, \Pi_{j=1}^{n-1} \sigma_j),$$

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This also shows how little on the quality of the Ritz value $\rho$ needs be said by

$$\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k} |e_k^T y|.$$

Any distance from $\rho$ to the spectrum of $A$ is possible with any value of $h_{k+1,k}$. 

Counterintuitive example 1: Convergence of interior Ritz values only:

\[ \mathcal{R} = \{ 3, \\
(3, 3), \\
(2, 3, 4), \\
(3, 3, 3, 3), \\
(1, 2, 3, 4, 5) \} . \]
Counterintuitive example 1: Convergence of interior Ritz values only:

\[ \mathcal{R} = \{ 3, (3, 3), (2, 3, 4), (3, 3, 3, 3), (1, 2, 3, 4, 5) \} . \]

This gives the unit upper Hessenberg matrix

\[
H(\mathcal{R}) = U(S)^{-1} C^{(5)} U(S) = \begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 1 \\
1 & 3 & -1 & 0 & 1 \\
1 & 3 & 5 & 0 & 1 \\
1 & 3 & 3 & 0 & 1 \\
\end{bmatrix}.
\]
2. Prescribed convergence for Arnoldi’s method

Thus these Ritz values are generated by the Arnoldi method applied to

\[
A = V \text{diag} \left( 1, \sigma_1, \ldots, \Pi_{j=1}^{n-1} \sigma_j \right) \begin{bmatrix}
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\end{bmatrix} \text{diag} \left( 1, \sigma_1, \ldots, \Pi_{j=1}^{n-1} \sigma_j \right)^{-1} V^*
\]

with initial vector \( v = V e_1 \) and for any unitary \( V \) and positive values \( \sigma_1, \ldots, \sigma_{n-1} \).
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with initial vector \( v = Ve_1 \) and for any unitary \( V \) and positive values \( \sigma_1, \ldots, \sigma_{n-1} \).

This is not a highly non-normal example, for instance with \( \sigma_i \equiv 1 \):

\[ \|A\|\|A^{-1}\| = 9.7137, \]

and the eigenvector basis \( W \) of \( A \) has condition number

\[ \|W\|\|W^{-1}\| = 4.8003. \]
Counterintuitive example 2: We can prescribe the “diverging” Ritz values

\[ \mathcal{R} = \{ 1, \]
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These Ritz values are generated by Arnoldi applied to

\[ A = VH(\mathcal{R})V^*, \quad v = Ve_1 \]

for unitary \( V \).
The same “diverging” Ritz values are generated with the exponentially decreasing values $2^{-1}$, $2^{-2}$, $2^{-3}$, and $2^{-4}$ on the subdiagonal of the Hessenberg matrix:

$$A = V \begin{bmatrix} 1 & 2 & 0 & -192 & 0 \\ 0.5 & 1 & 12 & 0 & -15872 \\ 0.25 & 1 & 48 & 0 \\ 0.125 & 1 & -160 \\ 0.0625 & 1 & 0 \end{bmatrix} V^*, \quad v = V e_1.$$
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Then the rounded residual norms $\|A(V_k y) - \rho(V_k y)\| = h_{k+1,k} |e_k^T y|$ seem to indicate convergence:

$$\left\{ \frac{1}{2}, \\
(0.1118, 0.1118), \\
(0.011, 0.0052, 0.011), \\
(0.0006, 0.0001, 0.0001, 0.0006) \right\}.$$
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Blue dots: Eigenvalues of $A$. Red dots: Ritz values in the first iteration.
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Blue dots: Eigenvalues of $A$. Red dots: Ritz values in the one but last iteration.
This negative result shows that no convergence result can be proved for the very popular Arnoldi method without special assumptions (see also [Embree - 2009] for the Arnoldi method with exact shifts).
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- This negative result shows that no convergence result can be proved for the very popular Arnoldi method without special assumptions (see also [Embree - 2009] for the Arnoldi method with exact shifts).

- Obviously, any Ritz value $\rho$ lies in the field of values of $A$:

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  H_k y = \rho y \quad \Rightarrow \quad y^* H_k y = \rho \quad \Rightarrow \quad y^* V_k^* A V_k y = \rho \quad (\|y\| = 1)
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The result therefore also shows how to construct matrices with a field of values containing $n(n + 1)/2$ prescribed complex points.
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- Thus in Crouzeix’s conjecture

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\|p(A)\| \leq 2 \max_{z \in W(A)} |p(z)|,
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the right-hand side is at least as large as

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- The following tries to gain insight in the conjecture for particular polynomials arising in Krylov subspace methods.
First, let us consider the GMRES polynomial.
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\[ Ax = b, \quad \|b\| = 1, \]

starting with initial guess \( x_0 = 0 \), GMRES iterates \( x_k \) minimize the residual vector \( r_k = b - Ax_k \):

\[ \|r_k\| = \|b - Ax_k\| = \min \|b - As\| \quad \text{over all} \quad s \in \mathcal{K}_k(A, b). \]
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The \( k \)th residual norm can be written as

\[ \|r_k\| = \|p^G_k(A)b\| = \min_{\pi \in \Pi^0_k} \|\pi(A)b\|, \]

where \( \Pi^0_k \) is the set of polynomials of degree at most \( k \) with the value one in the origin.
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The \( k \)th residual norm can be written as

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For the \( k \)th GMRES polynomial, Crouzeix’s conjecture is

\[ \|p_k^G(A)\| \leq 2 \max_{z \in \mathcal{W}(A)} |p_k^G(z)|. \]
3. Prescribed convergence for Arnoldi and GMRES

Obviously, 

\[ \| r_k \| = \| p_k^G (A) b \| \leq \| p_k^G (A) \| \leq 2 \max_{z \in W(A)} |p_k^G (z)|. \]
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Can we say anything about the relation between GMRES residual norms and \( |p_k^G(z)| \) on the field of values?
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We can say something about the relation between GMRES residual norms and at least the Ritz values: In general, there need not be any relation, they can be independent from each other.

Let us try to explain why: Writing \( x_k \) in the Arnoldi basis,

\[ x_k = V_k y_k \in \mathcal{K}_k(A, r_0), \]

and using the Arnoldi decomposition \( AV_k = V_{k+1} \tilde{H}_k \), we see that

\[ \| b - Ax_k \| = \| b - AV_k y_k \| = \| V_{k+1} e_1 - AV_k y_k \| \]
\[ = \| V_{k+1} (e_1 - \tilde{H}_k y_k) \| = \min_{y \in \mathbb{C}^k} \| e_1 - \tilde{H}_k y \|. \]
Thus the residual norms generated by the GMRES method are fully determined by the Hessenberg matrix $\tilde{H}_k$. 
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We have seen that the subdiagonal entries of $\tilde{H}_k$ can be chosen arbitrarily, for any prescribed Ritz values in the $k$th iteration.
3. Prescribed convergence for Arnoldi and GMRES

Thus the residual norms generated by the GMRES method are **fully determined** by the Hessenberg matrix $\tilde{H}_k$.

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- Hence there is a chance we can modify the behavior of GMRES while maintaining the prescribed Ritz values.
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- Hence there is a chance we can modify the behavior of GMRES while maintaining the prescribed Ritz values.

**Example from earlier:** Consider the prescribed 'diverging' Ritz values

$$\mathcal{R} = \{ 1, (0, 2), (-1, 1, 3), (-2, 0, 2, 4), (1, 1, 1, 1, 1) \},$$

and the prescribed subdiagonal entries of the generated Hessenberg matrix

$$\sigma_1 = 2^{-1}, \quad \sigma_2 = 2^{-2}, \quad \sigma_3 = 2^{-3}, \quad \sigma_4 = 2^{-4}.$$
3. Prescribed convergence for Arnoldi and GMRES

The corresponding GMRES convergence curve is

\[ \|r^{(0)}\| = 1, \quad \|r^{(1)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(2)}\| = \sqrt{\frac{1}{5}}, \quad \|r^{(3)}\| = 0.0052, \quad \|r^{(4)}\| = 0.0052. \]

Question: Can we force any GMRES convergence speed with arbitrary Ritz values by modifying the subdiagonal entries?

Not any, because there is a relation between GMRES stagnation and zero Ritz values: A singular Hessenberg matrix corresponds to stagnation in the parallel GMRES process, see [Brown - 1991]. In our example we have

\[ \rho_1^{(1)} = 1, \quad \|r^{(1)}\| = \frac{1}{\sqrt{5}} \]

\[ (\rho_1^{(2)}, \rho_2^{(2)}) = (0, 2), \quad \|r^{(2)}\| = \frac{1}{\sqrt{5}} \]

\[ (\rho_1^{(3)}, \rho_2^{(3)}, \rho_3^{(3)}) = (-1, 1, 3), \quad \|r^{(3)}\| = 0.0052 \]

\[ (\rho_1^{(4)}, \rho_2^{(4)}, \rho_3^{(4)}, \rho_4^{(4)}) = (-2, 0, 2, 4), \quad \|r^{(4)}\| = 0.0052. \]
3. Prescribed convergence for Arnoldi and GMRES

However, this is the only restriction Ritz values put on GMRES residual norms:

**Theorem 3** [DT, Meurant - 2012]. Consider a set of tuples of complex numbers

\[ R = \{ \rho_1^{(1)}, (\rho_1^{(2)}, \rho_2^{(2)}), \ldots, \rho_1^{(n-1)}, \ldots, \rho_{n-1}^{(n-1)}, (\lambda_1, \ldots, \lambda_n) \} , \]

such that \((\lambda_1, \ldots, \lambda_n)\) contains no zero number and \(n\) positive numbers

\[ 1 \geq f(1) \geq \cdots \geq f(n-1) > 0, \]

such that the \(k\)-tuple \((\rho_1^{(k)}, \ldots, \rho_k^{(k)})\) contains a zero number if and only if

\[ f(k-1) = f(k). \]
Let $A$ be a square matrix of size $n$ and let $b$ be a nonzero $n$-dimensional vector. The following assertions are equivalent:

1. The GMRES method applied to $A$ and right-hand side $b$ with zero initial guess yields residuals $\|r^{(k)}\| = f(k), \ k = 0, \ldots, n - 1$ such that

$$\|r^{(k)}\| = f(k), \ k = 0, \ldots, n - 1,$$

$A$ has eigenvalues

$$\lambda_1, \ldots, \lambda_n,$$

and

$$\rho_1^{(k)}, \ldots, \rho_k^{(k)}$$

are the Ritz values generated at the $k$th iteration for $k = 1, \ldots, n - 1$. 

3. Prescribed convergence for Arnoldi and GMRES
2. The matrix $A$ and right hand side $b$ are of the form

$$A = V U^{-1} C^{(n)} U V^*, \quad b = V e_1,$$

where $V$ is a unitary matrix,

$$U = \begin{bmatrix} g^T \\ 0 \\ T \end{bmatrix}$$

where the first row $g^T$ of $U$ is

$$g_1 = \frac{1}{f(0)}, \quad g_k = \sqrt{\frac{1}{f(k-1)^2} - \frac{1}{f(k-2)^2}}, \quad k = 2, \ldots, n.$$
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and the remaining submatrix $T$ of has entries satisfying

$$\prod_{i=1}^{k}(\lambda - \rho_i^{(k)}) = \frac{1}{t_{k,k}} \left( g_{k+1} + \sum_{i=1}^{k} t_{i,k} \lambda^i \right).$$
3. Prescribed convergence for Arnoldi and GMRES

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where the first row \( g^T \) of \( U \) is

\[
g_1 = \frac{1}{f(0)}, \quad g_k = \sqrt{\frac{1}{f(k-1)^2} - \frac{1}{f(k-2)^2}}, \quad k = 2, \ldots, n.
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and the remaining submatrix \( T \) of \( U \) has entries satisfying

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\prod_{i=1}^{k} (\lambda - \rho_i^{(k)}) = \frac{1}{t_{k,k}} \left( g_{k+1} + \sum_{i=1}^{k} t_{i,k} \lambda^i \right).
\]

Note we exhausted all freedom modulo unitary transformation.
3. Prescribed convergence for Arnoldi and GMRES

**Example:** Standardly converging Ritz values and 'nearly stagnating' GMRES:

\[ \mathcal{R} = \{ 5, (1, 5), (1, 4, 5), (1, 3, 4, 5), (1, 2, 3, 4, 5) \} \]

\[ \| r^{(0)} \| = 1, \quad \| r^{(1)} \| = 0.9, \quad \| r^{(2)} \| = 0.8, \]
\[ \| r^{(3)} \| = 0.7, \quad \| r^{(4)} \| = 0.6, \quad \| r^{(5)} \| = 0 \quad \text{gives} \]
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\[ A = V \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 10.3237 & 1 & 0 & 0 & 0 & 0 \\ 0.8458 & 4 & 0 & 0 & 0 & 0 \\ 3.312 & 3 & 0 & 0 & 0 & 0 \\ 2.4169 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} V^*, \quad b = Ve_1. \]
Again, this is not a highly non-normal example:

\[ \|A\| |A^{-1}| = 28.9498, \]

and the eigenvector basis \( W \) of \( A \) has condition number

\[ \|W\| |W^{-1}| = 57.735. \]

The residual norms \( \|A(V_ky) - \rho(V_ky)\| = h_{k+1,k} |e_k^T y| \) for the Ritz pairs are

10.3237,

(0.8458, 0.7886),

(0.8987, 3.312, 2.0509),

(0.9906, 2.4169, 2.3137, 1.7303).

respectively, i.e. they give misleading information.
Summarizing, any GMRES residual norms are possible with any Ritz values in all iterations.
3. Prescribed convergence for Arnoldi and GMRES

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- Any FOM residual norms are possible with any Ritz values in all iterations.

The FOM method differs from the GMRES method in that the residual norm is not minimized, but the $k$:th FOM residual vector is characterized through

$$r_k^F \perp \mathcal{K}_k(A, b).$$
The corresponding residual norms are related through to formula

\[
\frac{1}{\| r^F_k \|} = \sqrt{\frac{1}{\| r^G_k \|^2} - \frac{1}{\| r^G_{k-1} \|^2}}.
\]

Note that FOM residual norms need not be non-increasing and are not defined if the corresponding GMRES iterate stagnates.
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What seems interesting to me in the context of the Crouzeix’s conjecture is that the Ritz values are the roots of the FOM polynomials:

FOM polynomials might lead to a way to test if the conjecture can be disproved. For the \( k \)th FOM polynomial \( p_k^F \) we have,

\[ 0 = 2 \max_{\rho \text{ is a Ritz value}} |p_k^F(\rho)| < \|r_k^F\| = \|p_k^F(A)b\| \leq \|p_k^F(A)\|. \]
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The construction to prescribe Ritz values and FOM residual norms is the following:
3. Prescribed convergence for Arnoldi and GMRES

The matrix $A$ and right hand side $b$ are of the form

$$A = VU^{-1}C^{(n)}UV^*, \quad b = Ve_1,$$

where $V$ is a unitary matrix,

$$U = \begin{bmatrix} g^T \\ 0 \\ T \end{bmatrix}$$

where to force FOM residual norms $f(0), \ldots, f(n-1), f(i) > 0$, the first row $g^T$ of $U$ can be chosen as

$$g_k = \frac{1}{f(k-1)}, \quad k = 1, \ldots, n$$

and the remaining submatrix $T$ of has entries satisfying

$$\prod_{i=1}^{k} (\lambda - \rho_i^{(k)}) = \frac{1}{t_{k,k}} \left( g_{k+1} + \sum_{i=1}^{k} t_{i,k} \lambda^i \right).$$
Thank you for your attention.
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