The 2023 Alexanderson Award
Higher uniformity of bounded multiplicative functions

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The recipients of the fifth Alexanderson Award are Kaisa Matomäki, Maksym Radziwiłł, Terence Tao, Joni Teräväinen, and Tamar Ziegler for their paper "Higher uniformity of bounded multiplicative functions in short intervals on average" published in the Annals of Mathematics *in 2023. The award was given at the Joint Prize Ceremony at the 2024 Joint Mathematics Meeting in San Francisco. Joni Teräväinen delivered the Alexanderson Award Lecture "Uniformity of the Möbius Function in Short Intervals" on Thursday, January 4, 2024, and all five awardees co-organized an AIM Special Session on "Multiplicative Number Theory and Additive Combinatorics" at the JMM.*

This paper, which originated from a working group at the AIM workshop "Sarnak's conjecture" in December, 2018, focuses on the local Fourier uniformity conjecture for bounded multiplicative functions. The underlying principle behind the work is the belief that the additive and multiplicative structures of the integers are independent.

Rephrasing that idea more precisely involves the Liouville function λ. If you factor an integer into primes, $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then $\lambda(n) = (-1)^{e_1 + e_2 + \cdots + e_k}$. In other words, $\lambda(n)$ is 1 if n has an even number of prime factors, and -1 if n has an odd number of prime factors. The goal is to prove that the Liouville function is uncorrelated with a large class of sequences which arise in number theory, and also that it is uncorrelated with any nontrivial transformation of itself.

The pseudorandomness principle says that the Liouville function should behave like a random function $\mathbb{N} \to \{-1, 1\}$. That is, $\lambda(n)$ should take on the values 1 and -1 equally often, and furthermore there should be no patterns in the sequence of 1s and -1s. One way to measure how close $\lambda(n)$ is to random, is to look at the average

$$\langle \lambda(n) \rangle := \frac{1}{X} \sum_{n \leq X} \lambda(n).$$

The prime number theorem is equivalent to the statement that $\langle \lambda(n) \rangle = o(1)$ as $X \to \infty$. That statement is fairly weak: it says there is a lack of correlation between values of $\lambda(n)$ over long intervals $[1, X]$. The Riemann Hypothesis (RH) is equivalent to the statement that $\langle \lambda(n) \rangle = O_\epsilon(X^{-1/2+\epsilon})$ for any $\epsilon > 0$. Thus, RH asserts there is a lack of correlation over the much shorter interval $[1, X^{\frac{1}{2}+\epsilon}]$.

The conjectured randomness in $\lambda(n)$ is expressed in another way by Chowla's conjecture (related to the twin prime conjecture) which says that for distinct $h_1, h_2, \ldots, h_k$, we have $\langle \lambda(n+h_1)\lambda(n+h_2) \ldots \lambda(n+h_k) \rangle = o(1)$. In essence, the conjecture asserts that the Liouville function is uncorrelated with any nontrivial translation of itself.

Another consequence of the conjectured randomness of $\lambda(n)$ is that every possible length k sign pattern, $(\pm 1, \ldots, \pm 1)$, would occur infinitely many times as a value of $(\lambda(n+1), \lambda(n+2), \ldots, \lambda(n+k))$. In other words, if we let $s_k(\lambda)$ be the number of length k patterns which actually occur infinitely

many times, then $s_k(\lambda) = 2^k$. A weak version of Sarnak's conjecture says that λ is not deterministic; that is, $s_\lambda(k) \gg e^{ck}$ for some $c > 0$.

In recent years it has emerged that Chowla's conjecture and Sarnak's conjecture are deeply related, and a large amount of progress has been made on both in tandem. The awarded paper represents progress towards these two conjectures by proving, among other things, that $\tilde{s}_\lambda(k) \gg_A k^A$, where $\tilde{s}_\lambda$ counts the number of length-k patterns that occur at least once.

Some details of the mathematics in the paper and its consequences can be found on Terry Tao's blog post: `https://tinyurl.com/HigherUniformity`.