Rational approximations to $\zeta$

Keith Ball
This talk describes rational functions approximating $\zeta$:

\[
\frac{1}{(s-1)'} \quad \frac{s+1}{2(s-1)'} \quad \frac{4s^2 + 11s + 9}{6(s+3)(s-1)'} \quad \frac{(s+2)(3s^2 + 10s + 11)}{4(s^2 + 6s + 11)(s-1)'},
\]

\[
\frac{(s+2)(72s^3 + 490s^2 + 1193s + 1125)}{30(3s^3 + 29s^2 + 106s + 150)(s-1)'} \cdots
\]

The small size of $\zeta(1/2 + it)$ depends upon cancellation between different Dirichlet terms.

Each coefficient in the rational functions depends upon all the Dirichlet terms so the cancellation is built into the coefficients.
For each integer $m \geq 0$ we define

$$p_m(t) = (1 - t) \left(1 - \frac{t}{2}\right) \cdots \left(1 - \frac{t}{m}\right)$$

and the coefficients $(a_{m,j})$ by

$$p_m(t) = \sum_{0}^{m} (-1)^{j} a_{m,j} t^j.$$  

We then set

$$F_m(s) = \sum_{0}^{m} \frac{a_{m,j} B_j}{s + j - 1}$$

and

$$G_m(s) = \sum_{j=0}^{m} (-1)^{j} \frac{a_{m,j}}{s + j - 1}.$$
The rational functions in question are the ratios

\[
\frac{F_m(s)}{(s−1)G_m(s)}.
\]

For example

\[
F_3(s) = \frac{1}{s−1} - \frac{11}{12s} + \frac{1}{6(s+1)} = \frac{3s^2 + 10s + 11}{12(s−1)s(s+1)}
\]

and

\[
G_3(s) = \frac{1}{s−1} - \frac{11}{6s} + \frac{1}{s+1} - \frac{1}{6(s+2)} = \frac{s^2 + 6s + 11}{3(s−1)s(s+1)(s+2)}.
\]

The \(m^{\text{th}}\) ratio interpolates \(\zeta\) at the points \(0, −1, −2, \ldots, 1−m\) and has a simple pole with residue 1 at \(s = 1\).
The graph shows \((s - 1)\zeta(s)\) and the ratio \(F_5(s)/G_5(s)\)
The sequence converges locally uniformly to $\zeta$, at least to the right of the line $\{s: \Re s = 0\}$.

We shall see that

$$F_m(s) \approx h_m^{1-s} \Gamma(s) \zeta(s)$$

and

$$(s - 1)G_m(s) \approx h_m^{1-s} \Gamma(s)$$

where $h_m$ is the partial sum $\sum_{j=1}^{m} 1/j$ of the harmonic series.
The rational functions might still be difficult to analyse: what are the coefficients?

Focus on the $F_m$:

\[
\begin{align*}
F_0(s) &= \frac{1}{(s-1)}' \\
F_1(s) &= \frac{s + 1}{2(s - 1)s'} \\
F_2(s) &= \frac{4s^2 + 11s + 9}{12(s - 1)s(s + 1)}' \\
F_3(s) &= \frac{(s + 2)(3s^2 + 10s + 11)}{12(s - 1)s(s + 1)(s + 2)}
\end{align*}
\]

We have a recurrence relation: for each $m$

\[
(s + m - 1) F_m(s) = \frac{1}{(m + 1)} + (m + 1) \sum_{j=1}^{m} \frac{F_{m-j}(s)}{j(j + 1)}.
\]
Equivalently
\[
(1 + \frac{s - 1}{m}) F_m(s) = \frac{1}{m(m + 1)} + \frac{m + 1}{m} \sum_{j=1}^{m} \frac{F_{m-j}(s)}{j(j + 1)}.
\]

At each stage we take a weighted average of the previous terms, add a small bit and rotate slightly.

This is a very stable dynamical system.

The dependence of the end result $\zeta$ on $s$ can be very sensitive because $s$ rotates at each step. But for each fixed $s$ we have a very smooth way of getting to $\zeta(s)$. 
Here are the first few hundred values of $(n + 1)F_n(1/2 - 14i)$. 
If we treat the first $m + 1$ of these relations as a linear system for the values $F_0(s), F_1(s), \ldots, F_m(s)$ we can express the fact that $F_m(s) = 0$ by the vanishing of a certain determinant.

The numerator of the $m^{th}$ function is the determinant of

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots & 0 \\
\frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{m+1} & \frac{1}{m} & \frac{1}{m-1} & \cdots & \frac{1}{2} & 1
\end{pmatrix} + (1 - s) \begin{pmatrix}
0 & 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} \\
0 & 0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{m} \\
0 & 0 & 0 & \frac{1}{3} & \cdots & \frac{1}{m} \\
0 & 0 & 0 & 0 & \cdots & \frac{1}{m} \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]
So RH can be restated as what looks like a rather conventional spectral problem.

Connes reformulated RH as a statement about the spectrum of an operator acting on an infinite-dimensional function space.

There is a connection between Connes’ infinite-dimensional operator and these finite-dimensional ones.
If \( \Re s > 1 \)

\[
G_m(s) = \sum_{j=0}^{m} (-1)^j \frac{a_{m,j}}{s + j - 1} = \sum_{j=0}^{m} (-1)^j a_{m,j} \int_0^{1} x^j x^{s-2} \, dx \\
= \int_0^{1} p_m(x) x^{s-2} \, dx
\]

\[
p_m(x) = (1 - x) \left(1 - \frac{x}{2}\right) \ldots \left(1 - \frac{x}{m}\right) \approx e^{-h_m x}
\]

so it is no surprise that \( G_m(s) \approx h_m^{1-s} \Gamma(s - 1) \).

We want to do something similar for \( F_m \).
If $\Re s > 1$

\[
\int_0^\infty \frac{y}{1 - e^{-y}} e^{-y} y^{s-2} \, dy = \int_0^\infty \left( \sum_{n=1}^\infty e^{-ny} \right) y^{s-1} \, dy \\
= \sum_{n=1}^\infty \int_0^\infty e^{-ny} y^{s-1} \, dy = \sum_{n=1}^\infty \frac{1}{n^s} \Gamma(s).
\]

So

\[
\Gamma(s) \zeta(s) = \int_0^\infty \frac{-\log(1 - (1 - e^{-y}))}{1 - e^{-y}} e^{-y} y^{s-2} \, dy \\
= \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1} (1 - e^{-y})^k e^{-y} y^{s-2} \, dy.
\]
\[ \Gamma(s) \zeta(s) = \int_0^\infty \sum_{k=0}^\infty \frac{1}{k+1} (1 - e^{-y})^k e^{-y} y^{s-2} \, dy. \]

Using a standard formula for Bernoulli numbers we get that for \( \Re s > 1 \)

\[ F_m(s) = \int_0^1 \left( \sum_{k=0}^m \frac{1}{k+1} \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \right) x^{s-2} \, dx \]

If \( x \) is close to zero then

\[ \Delta_{m,k}(x) = \sum_{r=0}^k \binom{k}{r} (-1)^r p_m((r+1)x) \approx \sum_{r=0}^k \binom{k}{r} (-1)^r e^{-h_m(r+1)x} = (1 - e^{-h_{mx}})^k e^{-h_{mx}}. \]
For small values of $x$ the integrand is approximately
\[
\left( \sum_{k=0}^{m} \frac{1}{k+1} e^{-hm x} (1 - e^{-hm x})^k \right) x^{s-2}.
\]

If the approximation were good for all $x$ between 0 and 1 then $F_m(s)$ would be close to
\[
\int_0^1 \sum_{k=0}^{m} \frac{1}{k+1} e^{-hm x} (1 - e^{-hm x})^k x^{s-2} dx
\]
\[
= h_m^{1-s} \int_0^{hm} \sum_{k=0}^{m} \frac{1}{k+1} e^{-y} (1 - e^{-y})^k y^{s-2} dy
\]

and the integral converges to $\Gamma(s)\zeta(s)$ as $m \rightarrow \infty$. 
We want to show that

\[ h_m^{s-1} F_m(s) \to \Gamma(s)\zeta(s) \]

locally uniformly for \( \Re s > 0 \).

Crossing the pole at \( s = 1 \) is not the problem.

The difficulty is that unless \( x \) is very close to 0, the expressions

\[ \Delta_{m,k}(x) = \sum_{r=0}^{k} \binom{k}{r}(-1)^r p_m((r+1)x) \]

involve values of \( p_m \) at points well outside the interval \([0, 1]\).
The graph shows the $\Delta_{m,k}$ for $m = 10$. 
Lemma 1 (Key Lemma). If $m$ is a non-negative integer, $k$ is any integer and $x \in [0, 1]$ then

$$\Delta_{m,k}(x) \geq 0.$$ 

It is trivial to check that

$$\sum_{k=0}^{m} \Delta_{m,k}(x) = 1$$

for all $x$, so the $\Delta_{m,k}$ form a partition of unity on $[0, 1]$. 
After some fairly delicate estimates we get that the ratios
\[
\frac{F_m(s)}{(s - 1)G_m(s)}
\]
converge locally uniformly to \(\zeta(s)\) for \(\Re s > 0\).

My guess is that they do so on the entire complex plane.

**Theorem 2** (*Convergence*).
\[
h_m^{s-1}(s - 1)F_m(s) \to (s - 1)\Gamma(s)\zeta(s)
\]
locally uniformly for \(\Re s > 0\).
Lemma 1 (Key Lemma). If $m$ is a non-negative integer, $k$ is any integer $k$ and $x \in [0, 1]$

$$\Delta_{m,k}(x) \geq 0.$$ 

The proof of the key lemma involves the introduction of an additional parameter. For each $v$ define

$$P_m(v, x) = (v + 1 - x)(v + 2 - x) \ldots (v + m - x)$$

and

$$\tilde{\Delta}_{m,k}(v, x) = \sum_{r=0}^{k} \binom{k}{r} (-1)^r P_m(v, (r + 1)x).$$

$$\tilde{\Delta}_{m,k}(0, x) = m! \Delta_{m,k}(x)$$ so the key lemma follows from:
**Lemma 3.** If $m$ is a non-negative integer, $k$ is an integer, $v \geq 0$ and $0 \leq x \leq 1$ then

$$\tilde{\Delta}_{m,k}(v, x) \geq 0.$$ 

*Proof* We use induction on $m$. When $m = 0$, $\tilde{\Delta}_{m,k}(v, x)$ is zero unless $k = 0$ in which case it is 1.

We claim that for $m > 0$

$$\tilde{\Delta}_{m,k}(v, x) = (v + 1 - x)\tilde{\Delta}_{m-1,k}(v + 1, x) + kx\tilde{\Delta}_{m-1,k-1}(v + 1 - x, x).$$

Then the inductive step is clear because we can assume that $k \geq 0$ and for the given range of $v$ and $x$, the number $v + 1 - x$ is also at least 0.

\[\Box\]
Estimating the size of $\zeta$

We have that

$$F_m(s) = \int_0^1 f_m(x)x^{s-2} \, dx$$

where

$$f_m(x) = \sum_{k=0}^m \frac{1}{k+1} \Delta_{m,k}(x).$$

Numerical evidence indicates that the function $f_m(x/h_m)$ differs from $x/(e^x - 1)$ by only about $h_m/m$ at any point of $[0, h_m]$ and so we expect the ratio

$$\frac{F_m(s)}{(s-1)G_m(s)}$$

to provide a good approximation to $\zeta$ at $s = 1/2 + it$ as long as $\Gamma(s)$ is as large as $h_m/m$. 
We expect the ratio

\[ F_m(s) \over (s - 1)G_m(s) \]

to provide a good approximation to \( \zeta \) at \( s = 1/2 + it \) as long as \( \Gamma(s) \) is as large as \( h_m/m \).

This happens if \(|t|\) is at most a bit less than \( {2 \over \pi} \log m \).

Rough calculations indicate that the ratio is not too far from \( \zeta \) for \( t \) all the way up to \( \log m \).

There are good reasons to think that \( F_m(s) \) does not oscillate significantly for \( t \) larger than \( \log m \).
The connection with Connes’ operator

The Toeplitz matrix

\[
L_m = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\
\frac{1}{3} & \frac{1}{2} & 1 & 0 & \cdots & 0 \\
\frac{1}{4} & \frac{1}{3} & \frac{1}{2} & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{m+1} & \frac{1}{m} & \frac{1}{m-1} & \cdots & \frac{1}{2} & 1
\end{pmatrix}
\]

can be thought of as acting on polynomials \(a_0 + a_1x + a_2x^2 + \cdots + a_mx^m\) rather than sequences \((a_0, \ldots, a_m)\).
It does so by multiplication by the partial sum
\[ \sum_{0}^{m} \frac{x^j}{j + 1} \]
of the series for \( \frac{-\log(1-x)}{x} \) (followed by truncation back to a polynomial of degree \( m \)).

In this context the upper triangular matrix \( U_m \) maps a polynomial \( q \) of degree \( m \) to
\[ \frac{1}{1 - x} \int_{x}^{1} \frac{q(t) - q(0)}{t} \, dt. \]

The operator of Connes is built from a multiplication operator and an integral operator much like these, acting on an infinite-dimensional function space.