PSEUDO-LAPLACIANS: A SPECIAL CASE

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DeVerdière tentative conjecture, 1983

In his pioneering paper: Yves Colin de Verdière, Pseudo-laplaciens II, *Annales de l'institut Fourier*, tome 33 (1983), 87-113, the author formulated a somewhat tentative conjecture stating that the zeros s_j of the function

$$\int_0^\infty \frac{\left|E\left(\rho, \frac{1}{2}+ir\right)\right|^2}{s_j(1-s_j)-(\frac{1}{4}+r^2)}\,\mathrm{d}r$$

where $E(\rho, s)$ is the Eisenstein series associated to the cubic root of unity ρ, are precisely the zeros of the Eisenstein series. He proved some numerical lower bounds consistent with the conjecture.

In what follows, we write $E_s(z)$ for the Eisenstein series, always using the subscript symbol for the complex variable, while the variable in parentheses is the parameter in the fundamental domain.

Numerics of zeros of CdV functional

$\zeta(s)L(s,\chi_{-3})=0$	zeros of CdV functional
	7.01
8.03973715568143	8.019
11.24920620777292	11.072
14.13472514173469	14.070
15.70461917672160	?
18.26199749569307	$[18.0, 18.1]$
20.45577080774248	?
21.02203963877155	?
24.05941485649342	$[24.0, 24.01]$
25.01085758014568	?
26.57786873577453	?
28.21816450623334	?
30.42487612585952	?

Old computation. Much better data are available elsewhere.

Heegner distributions

a) $d < 0$ a discriminant (not necessarily a fundamental discriminant)

b) $h'(d)$ is the number of Lagrange reduced quadratic forms $Ax^2 + Bxy + Cy^2$ of discriminant $d = B^2 - 4AC$ weighted by their Heegner points, namely:

Heegner points $\mathfrak{z} = \frac{-B+\sqrt{d}}{2A} \in \Gamma\backslash\mathfrak{H}$, counted with weight $w(\mathfrak{z}) = 1$

(but $w(\rho) = 1/3$ and $w(i) = 1/2$)

c) the Heegner distribution θ_d and the Hirzebruch–Zagier modified class number $h'(d)$ are given by

$$\theta_d = \sum_{\mathfrak{z} \in H_d} w(\mathfrak{z}) \delta_{\mathfrak{z}}^{nc}, \quad h'(d) = \sum_{\mathfrak{z} \in H_d} w(\mathfrak{z})$$

Constant term distributions and Eisenstein series

The constant term distribution at height $a > 0$ is by definition

$$\eta_a f = \int_0^1 f(a + ix)dx.$$

It is a compactly supported, real-valued, regular Borel measure on $\Gamma\backslash\mathfrak{H}$ and a continuous functional on $C^0(\Gamma\backslash\mathfrak{H})$.

A pseudo-Eisenstein series is the sum of all translates by Γ of a smooth function on $(0,\infty)$ with compact support:

$$\Psi_\varphi(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \varphi(\Im(\gamma z)) \quad (\varphi \in C_c^\infty(0,\infty)).$$

A classical Eisenstein series is

$$E_s(z) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \Im(\gamma z)^s \qquad (\Gamma_\infty = \text{parabolic stabilizer of the cusp } i\infty)$$

and by analytic continuation for general s.

The zeta function of orders in imaginary quadratic fields, I

Let $d = d_o f^2$ where d_0 is a fundamental discriminant and let $\mathfrak{O}_d$ be the order of discriminant d in the ring of integers of the field $Q(\sqrt{d_0})$. Then

$$\sum_{y \in \mathfrak{Z}_d} w(\mathfrak{z}) E_s(\mathfrak{z}) = \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s, \mathfrak{O}_d)}{\zeta(2s)}$$

where $\mathfrak{O}_d$ is the order of discriminant d in the ring of integers of the field $\mathbb{Q}(\sqrt{d_0})$. This again has an Euler product, as one sees from

$$\sum_{\mathfrak{z} \in \mathfrak{Z}_d} w(\mathfrak{z}) E_s(\mathfrak{z}) = \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s, \mathfrak{O}_d)}{\zeta(2s)}$$

$$= \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s)\, L(s, \chi_{d_0})}{\zeta(2s)} \sum_{\delta m k | f} \frac{\mu(\delta)\mu(m)\chi_{d_0}(m) k}{(\delta m k^2)^s}.$$

The important fact is that the zeta function of the order is divisible by $\zeta(s)$ and $L(s, \chi_{d_0})$. This is a really amazing property, yielding the existence of infinitely many finite linear combinations of Eisenstein series having infinitely many non-trivial zeros in common.

The zeta function of orders in imaginary quadratic fields, II

By $\mathcal{D}$ and the *weight* $W(\mathcal{D})$, we mean:

- A set $\mathcal{D}$ of discriminants $d < 0$, with associated fundamental discriminant d_0, hence $d = d_0 f^2$, with $d_0 = (-1, -4, -8) \times \{$odd squarefree number$\}$
- For each discriminant $d \in \mathcal{D}$, we have the Heegner set $\mathfrak{Z}_d$ of Heegner points $\mathfrak{z}$, each taken with weight $\nu_d w(\mathfrak{z})$, and associated distribution $\nu_d \theta_d$
- The weight $W(\mathcal{D})$ of $\mathcal{D}$ is given by:

$$W(\mathcal{D}) = \sum_{(d,\nu_d) \in \mathcal{D}} \nu_d h'(d).$$

We refer to the set of triples $\{d, \theta_d, \nu_d\}$ as a *complete Heegner set*.

Spectral decomposition and spectral synthesis

The *spectral transform* $f \to \mathcal{E}f$ of a pseudo-Eisenstein series is

$$\mathcal{E}f(s) = \int_{\Gamma\backslash\mathfrak{H}} f(z) E_{1-s}(z)\ \mathrm{d}\omega_z$$

with $\mathrm{d}\omega_z = y^{-2}\ \mathrm{d}x\,\mathrm{d}y$ the hyperbolic area element at z.

At least pointwise, we have convergence of the *spectral synthesis* for f in the closure of the space of pseudo-Eisenstein series, namely:

$$f(z) = \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \mathcal{E}f(s) \cdot E_s(z)\ \mathrm{d}s \qquad (\text{for } f \in D)$$

where $\int_{(\frac{1}{2})}$ is integration along the vertical line $\Re(s) = \frac{1}{2}$.

The spectral synthesis of the non-cuspidal Dirac functional

The spectral expansion of the non-cuspidal Dirac functional is

$$\delta_{z_0}^{nc} = \frac{1}{\langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} E_{1-s}(z_0) \cdot E_s \, \mathrm{d}s$$

hence

$$\mathcal{E}_{\delta_{z_0}^{nc}}(s) = E_{1-s}(z_0).$$

Thus $\delta_{z_0}^{nc} f = f(z_0)$, as expected.

The spectral synthesis of θ_d

We denote by χ_d the quadratic character determined by the Kronecker symbol $(d/\cdot)$. This is a primitive character if and only if d is a fundamental discriminant. In every case we have

$$\mathcal{E}\theta_d = \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s, \mathfrak{O}_d)}{\zeta(2s)}.$$

By linearity, this extends to $\mathcal{D}$ in place of d and the condition $W(\mathcal{D}) = 0$ ensures the orthogonality property

$$\langle \theta_{\mathcal{D}}, 1 \rangle = 0$$

and $\theta_{\mathcal{D}} \in V^{\perp 1}_{-1-\varepsilon}$. We have the spectral expansion

$$\theta_{\mathcal{D}} = \frac{W(\mathcal{D}) \cdot 1}{\langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} E_{1-s}(\mathcal{D}) \cdot E_s \, \mathrm{d}s$$

and the functional equation $E_s(\mathcal{D}) = c_s E_{1-s}(\mathcal{D})$.

Recall of basic notation

We define the r^{th} weighted L^2 norm $|\cdot|_{X_r}$ on $\mathcal{E}D^{\perp 1}$ by

$$|\mathcal{E}f|^2_{X_r} := \frac{1}{4\pi i}\int_{(\frac{1}{2})} |\mathcal{E}f(s)|^2\,\lambda_s^r\,\mathrm{d}s \qquad\qquad (f \in D^{\perp 1},\ \lambda_s = s(1-s))$$

The corresponding *Sobolev norm* on $D^{\perp 1}$ is

$$|f|^2_r := |\mathcal{E}f|^2_{X_r}.$$

and

$$X_r = \text{completion of } \mathcal{E}D^{\perp 1} \text{ with respect to } |\cdot|_{X_r}.$$

We also define

$$V_r^{\perp 1} = \text{completion of } D^{\perp 1} \text{ with respect to } |\cdot|_r\,, \quad V_r = \mathbb{C} \oplus V_r^{\perp 1}.$$

Main properties of Eisenstein series

Here S is $S = -\Delta$ with Δ the hyperbolic Laplacian.

$$(S + \lambda_s)E_s(z) = 0, \qquad E_s(z) = c_s E_{1-s}(z)$$

with c_s given by

$$c_s = \sqrt{\pi}\,\frac{\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\Gamma(s)\zeta(2s)} = \frac{\xi(2-2s)}{\xi(2s)},$$

where in the last step $\xi(s)$ is the completed Riemann zeta function. This yields

$$c_s c_{1-s} = 1, \qquad c_{\frac{1}{2}} = -1, \qquad E_{\frac{1}{2}}(z) = 0.$$

The Eisenstein series $E_s(z)$ is not in L^2 because

$$y^s + c_s y^{1-s} = \int_0^1 E_s(x+iy)\,\mathrm{d}x,$$

yielding a logarithmic divergence of the L^2 norm at the cusp.

Solving $(-\Delta - \lambda_w)u = \theta_{\mathcal{D}}$ and $(-\Delta - \lambda_w)u = \eta_a$

For $\Re(w) > \frac{1}{2}$, the equation $(-\Delta - \lambda_w)u = \theta_{\mathcal{D}}$ has an unique solution $u_{\mathcal{D},w}$, in fact in $V_{\frac{3}{2}-\varepsilon}$ for $\varepsilon > 0$, with spectral expansion

$$u_{\mathcal{D},w} = \frac{W(\mathcal{D}) \cdot 1}{(\lambda_1 - \lambda_w) \cdot \langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} E_{1-s}(\mathcal{D}) \cdot E_s \frac{\mathrm{d}s}{\lambda_s - \lambda_w}$$

with $W(\mathcal{D})$ the weight of $\mathcal{D}$.

For $\Re(w) > \frac{1}{2}$ the equation $(-\Delta - \lambda_w)u = \eta_a$ has an unique solution $v_{w,a} \in V_{\frac{3}{2}-\varepsilon}$ for all $\varepsilon > 0$, with spectral expansion

$$v_{w,a} = \frac{1}{(\lambda_1 - \lambda_w) \cdot \langle 1, 1 \rangle} + \frac{1}{4\pi i} \int_{(\frac{1}{2})} (a^{1-s} + c_{1-s} a^s) \cdot E_s \frac{\mathrm{d}s}{\lambda_s - \lambda_w}.$$

Solving $\widetilde{S}u = \lambda_w u$ with a certain Friedrichs extension $\widetilde{S}$

Here $\widetilde{S}$ is the Friedrichs extension of S which "ignores" the 2-dimensional space $\Theta = \ker(\theta_{\mathcal{D}} \oplus \eta_a)$, which plays the role of a "boundary condition".

Theorem *The condition for the existence of a non-zero solution*

$$u = z_1 u_{\mathcal{D},w} + z_2 v_{w,a}$$

of $\widetilde{S}u = \lambda_w u$ is the vanishing of the determinant

$$\det \begin{pmatrix} \theta_{\mathcal{D}}(u_{\mathcal{D},w}) & \theta_{\mathcal{D}}(v_{w,a}) \\ \\ \eta_a(u_{\mathcal{D},w}) & \eta_a(v_{w,a}) \end{pmatrix} = 0.$$

Computing $\eta_a(v_{w,a})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

The computation of $\eta_a(v_{w,a})$ is quite easy from the spectral expansion:

$$\eta_a(v_{w,a}) = \frac{1}{(\lambda_1 - \lambda_w)\cdot\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s} + c_{1-s}a^s)(a^s + c_s a^{1-s})\frac{\mathrm{d}s}{\lambda_s - \lambda_w}$$

$$= \frac{1}{(\lambda_1 - \lambda_w)\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a + c_{1-s}a^{2s} + c_s a^{2-2s} + a)\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \quad \text{(use } c_s c_{1-s} = 1\text{)}$$

$$= \frac{1}{(\lambda_1 - \lambda_w)\langle 1,1\rangle} + \frac{1}{2\pi i}\int_{(\frac{1}{2})} (a + c_s a^{2-2s})\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \quad (s \to 1-s \text{ in one term}).$$

By moving the line of integration to $+\infty$ one finds

$$\eta_a(v_{w,a}) = -\frac{a}{w-(1-w)} - \frac{c_w a^{2-2w}}{w-(1-w)}$$

$$= \frac{a + c_w a^{2-2w}}{1-2w} \qquad (a > 1,\ \Re(w) > \tfrac{1}{2}).$$

Computing $\theta_{\mathcal{D}}(v_{w,a})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

$$\begin{aligned}\delta_z^{\mathsf{nc}}(v_{w,a}) &= \frac{1}{(\lambda_1 - \lambda_w)\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} \eta_a E_{1-s}(z)\cdot E_s(z)\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\ &= \frac{1}{(\lambda_1 - \lambda_w)\langle 1,1\rangle} + \frac{1}{4\pi i}\int_{(\frac{1}{2})} (a^{1-s} + c_{1-s}a^s)\cdot E_s(z)\frac{\mathrm{d}s}{\lambda_s - \lambda_w} \\ &= \frac{1}{(\lambda_1 - \lambda_w)\langle 1,1\rangle} + \frac{1}{2\pi i}\int_{(\frac{1}{2})} a^{1-s}E_s(z)\frac{\mathrm{d}s}{\lambda_s - \lambda_w}. \end{aligned} \tag{1}$$

The computation of the integral requires some extra care, which depends on the height of z relative to a. To this end, we proceed as before moving the line of integration from $\sigma = \frac{1}{2}$ to $\sigma = C$ where $C > 1$, thereby acquiring the contribution of residues at $s = w$ and also at $s = 1$ from the Eisenstein series.

Computing $\theta_{\mathcal{D}}(v_{w,a})$, continued

This yields

$$\delta_z^{\mathsf{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{1-2w} + \frac{1}{2\pi i}\int_{(C)} a^{1-s}E_s(z)\frac{\mathrm{d}s}{\lambda_s - \lambda_w}$$

The series for $E_s(z) = \sum' y^s/|mz+n|^{2s}$ with $z = x + iy$ is absolutely convergent for $\Re(s) = c > 1$ and for $y \to \infty$ it is asymptotic to y^s. If $y/a < 1$ one may move the line of integration all the way to $C = +\infty$, showing that the integral in question vanishes.

If instead $y/a > 1$, only the finitely many terms with $|mz+n|^2 \leqslant y/a$ contribute to the integral. In fact, in this case it must be that $m = 0$ and $n = \pm 1$. Then one moves the line of integration backwards all the way to $-\infty$, encountering two residues at $s = w$ and $s = 1 - w$ and with the limit of the integral being 0. The final result is

$$\delta_z^{\mathsf{nc}}(v_{w,a}) = \frac{a^{1-w}E_w(z)}{1-2w} - \frac{a^{1-w}y^w - a^w y^{1-w}}{1-2w}.$$

Computing $\theta_{\mathcal{D}}(v_{w,a})$, end

Theorem *Let $a > 1$, $\Re(w) > \frac{1}{2}$ and assume that a is not equal to the imaginary part of any Heegner point occurring in $\mathcal{D}$. Then*

$$\theta_{\mathcal{D}}(v_{w,a}) = \frac{1}{1-2w}\left\{a^{1-w}E_w(\mathcal{D}) - R_w(\mathcal{D},a)\right\}$$

where we have set

$$R_w(\mathcal{D},a) = \sum_d \nu_d \sum_{\substack{x+iy\in\mathfrak{Z}_d \\ y>a}} (a^{1-w}y^w - a^w y^{1-w}).$$

Computing $\eta_a(u_{\mathcal{D},w})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

Theorem $\eta_a(u_{\mathcal{D},w}) = \theta_{\mathcal{D}}(v_{w,a})$.

Computing $\theta_{\mathcal{D}}(u_{\mathcal{D},w})$ for $a > 1$ and $\Re(w) > \frac{1}{2}$

Theorem *If $W(\mathcal{D}) = 0$ then*

$$\theta_{\mathcal{D}}(u_{\mathcal{D},w}) = \frac{1}{4\pi i}\int_{(\frac{1}{2})} |E_s(\mathcal{D})|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w}.$$

The resolvent

Theorem *For all* $a > 1$, *all* w *with* $\frac{1}{2} < \Re(w) < 1$ *and off* $(\frac{1}{2}, 1)$, *and all* $\mathcal{D}$ *with* $W(\mathcal{D}) = 0$, *it holds*

$$\frac{a + c_w a^{2-2w}}{1-2w} \frac{1}{4\pi i} \int_{(\frac{1}{2})} |E_s(\mathcal{D})|^2 \frac{\mathrm{d}s}{\lambda_s - \lambda_w} - \frac{1}{(1-2w)^2} \Big(a^{1-w} E_w(\mathcal{D}) - R_w(\mathcal{D}, a)\Big)^2 \neq 0$$

where

$$E_s(\mathcal{D}) = \sum_{d \in \mathcal{D}} \nu_d \left(\frac{\sqrt{|d|}}{2}\right)^s \frac{\zeta(s, \mathfrak{O}_d)}{\zeta(2s)}$$

and where

$$R_w(\mathcal{D}, a) = a \sum_d \nu_d \sum_{\substack{x+iy \in \mathfrak{Z}_d \\ y > a}} \Big((y/a)^w - (y/a)^{1-w}\Big).$$

Proof Since the operator is self-adjoint, any eigenvalue $w(1-w)$ must be real and positive.

The average of zeta-functions of orders of quadratic fields

No matter the choice of $\mathcal{D}$, the function $E_s(\mathcal{D})$ is divisible by $\zeta(s)/\zeta(2s)$. The average of zeta-functions of orders was done by A.I. Vinogradov and Thaktadzhyan in 1981. Here it is (with our notation):

Theorem *Let $\mathcal{D}$ be the set of all negative discriminants of absolute value up to D, all of them counted with weight $\nu_d = 1$. Let $s = \sigma + it$, $0 \leqslant \sigma \leqslant 1$, $\varepsilon > 0$, and t fixed. Then as $D \to \infty$ it holds*

$$E_s(\mathcal{D}) = \Phi(s)D^{1+s/2} + c_s\Phi(1-s)D^{1+(1-s)/2} + W_s(D)$$

where

$$\Phi(s) = \frac{2^{-s}\zeta(s)}{(s+2)\zeta(s+2)}$$

and $W_s(D) = O\Big(|\zeta(2s)|^{-1}(1+|c_s|)D^{\frac{3}{4}+\varepsilon}\Big)$.

The asymptotic evaluation of $R_w(\mathcal{D}, a)$ for fixed w

We take $\mathcal{D} = \{D/K^2 < |d| \leqslant D\}$ where $K \to \infty$ at a suitably slow rate and split the interval into two parts, each with constant weight chosen so to satisfy $W(\mathcal{D}) = 0$. An immediate appeal to the well-known Perron formula for estimating a partial sum of a Dirichlet series fails, because the range of summation depends on $|d|$. Moreover, there is no smoothing of the sum and the last term can play a significant role. So, the pedestrian way was to apply the Perron formula to each sum, averaging the individual results. After two weeks, the conclusion was only a lemma:

Lemma *Let $w = u + iv$ and assume $0 < u < 1$. Let $D/K^2 < D^* \leqslant D$. Then for $\varepsilon > 0$ it holds*

$$\sum_{|d| \leqslant D^*} \left(\frac{\sqrt{|d|}}{2} \right)^w \sum_{A < \sqrt{|d|/D}\,K} \frac{b(d, A)}{A^w} = \frac{1}{3\zeta(3)} \frac{K^{1-w}}{1-w} D^{1+\frac{w}{2}} \left(\frac{D^*}{D} \right)^{\frac{3}{2}} + O\left(K^{\max(\frac{1}{2} - u, 0) + \varepsilon} D^{1+\frac{u}{2}} \left(\frac{D^*}{D} \right)^{\frac{3}{2}} \right)$$

The smart evaluation of $R_w(\mathcal{D}, a)$ for fixed w

The smart evaluation of the sum was done by Henryk Iwaniec in just two hours (not two weeks). Iwaniec's evaluation of $R_w(a, \mathcal{D})$ yields a precise asymptotic formula:

Theorem (Iwaniec) *Split the interval* $[D/K^2, D]$ *into two subintervals* $\Delta_1 \cup \Delta_2$ *at the point* $D_1 = \alpha D$ *with* $0 < \alpha < 1$ *fixed, taking weights* $\nu_d = \tau < 0$ *on* Δ_1, $\nu_d = 1$ *on* Δ_2.

Then we have

$$R_w(\mathcal{D}) = Q(w; D, K) - Q(1-w; D, K) + O\Big((1+|w|)K^{-1}D^{\frac{3}{2}}\Big)$$

with

$$Q(w; D, K) = \frac{K^{w-1}\zeta(w)}{4(w+2)\zeta(w+2)}\Big(1-(1-\tau)\alpha^{1+\frac{w}{2}}\Big)D^{\frac{3}{2}}.$$

Note: The condition $W(\mathcal{D}) = 0$ is not needed here.

Note: Both $Q(w; D, K)$ and $Q(1-w; D, K)$ vanish when w is a non-trivial zero of the zeta function.

Evaluation of $E_w(\mathcal{D}, a)$ for fixed w with $\Re(w) > \frac{1}{2}$

Here, w belongs to any fixed compact set in the open infinite strip $\frac{1}{2} < \Re(w) < 1$. The evaluation follows immediately from the Vinogradov–Tathkadzhyan theorem:

Theorem *Split the interval* $[D/K^2, D]$ *into two subintervals* $\Delta_1 \cup \Delta_2$ *at the point* $D_1 = \alpha D$ *with* $0 < \alpha < 1$ *fixed and take weights* $\nu_d = \tau < 0$ *on* Δ_1, $\nu_d = 1$ *on* Δ_2, *satisfying the condition* $\sum_d \nu_d h'(d) = 0$.
Then with these weights, $a = \sqrt{D}/(2K)$, *and* $\frac{1}{2} < \Re(w) < 1$, *it holds*

$$a^{1-w} E_w(\mathcal{D}) = \frac{K^{w-1}\zeta(w)}{2(w+2)\zeta(w+2)} D^{\frac{3}{2}} \left(1 - (1-\tau)\alpha^{1+\frac{w}{2}} + O(K^{-2})\right).$$

This asymptotic evaluation is uniform in w only for w in any fixed compact subset of the open vertical strip $\frac{1}{2} < \Re(w) < 1$.

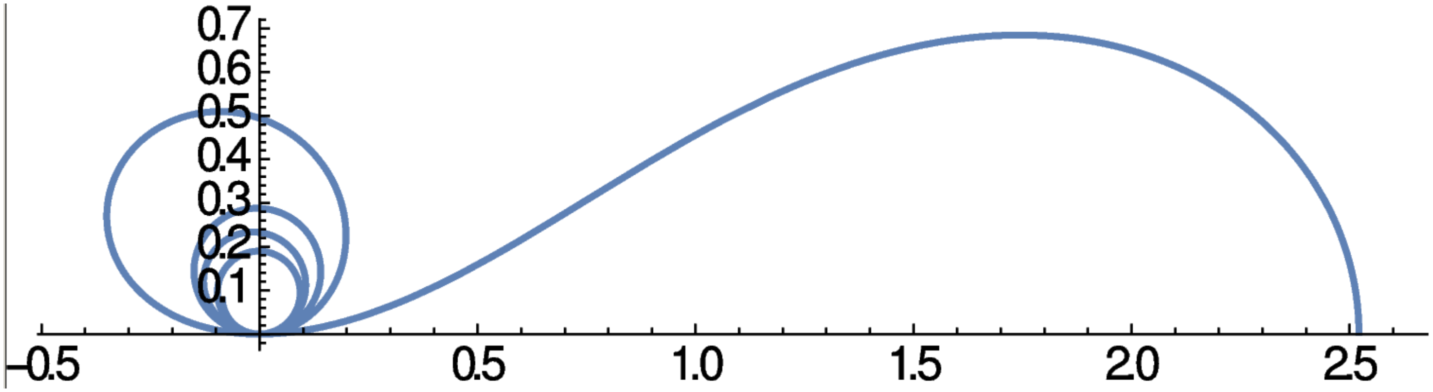
The weighted L^2 mean of $E_s(\mathcal{D}, a)$ when $\Re(s) = \frac{1}{2}$

The evaluation of the left-hand side of the resolvent is not yet completed at this time. By symmetry, it suffices to consider only the integral on the half-line $\Re(s) = \frac{1}{2}$, $\Im(s) \geqslant 0$. The Vinogradov–Tathkadzhyan theorem shows immediately that the integral over an initial segment $[0, T_0]$ with $T_0 = o(\log D)$ is of precise order $D^{\frac{5}{2}}$.

However, the Vinogradov–Tathkadzhyan estimate fails completely when T_0 is large and one can show, somewhat indirectly, that the correct order of the weighted L^2-mean is actually of order $T^{\frac{5}{2}}(\log T)^{A+o(1)}$ for some strictly positive constant A. As yet, we do not know the exact asymptotics in question and it presents an interesting question for the analytic number theorist.

This can be seen as follows.

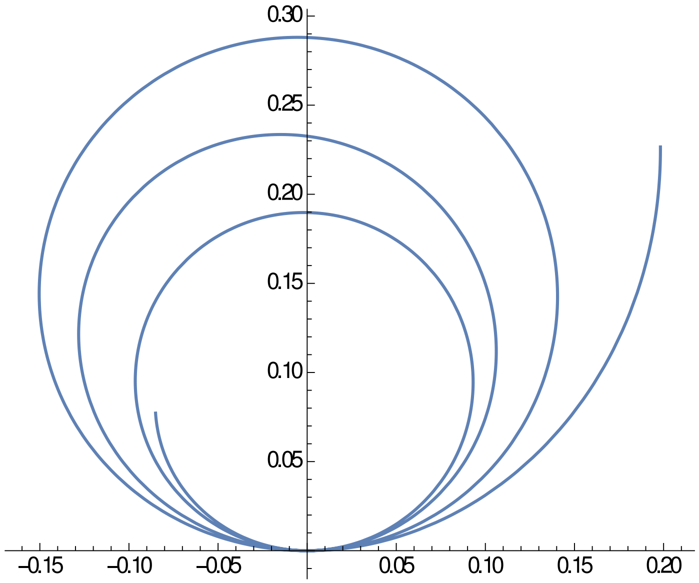
A picture of $(a + c_w a^{2-2w})/(1-2w)$



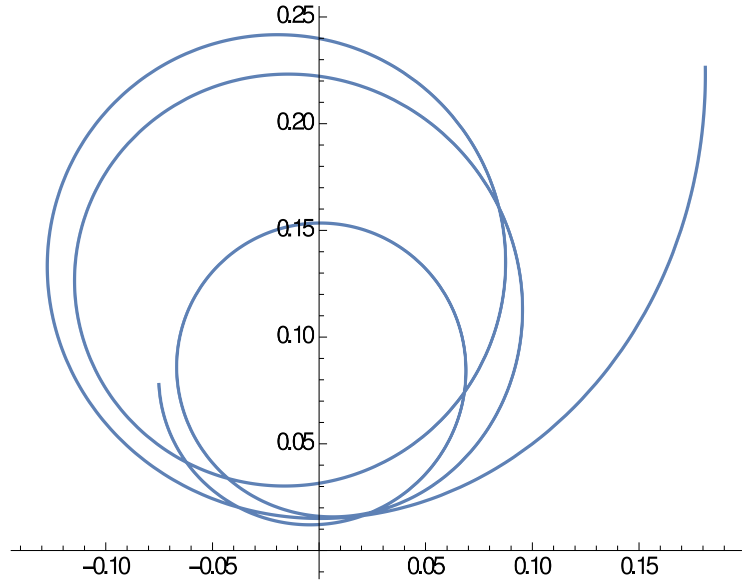
all points $\dfrac{a + c_w a^{2-2w}}{1-2w}$ when $a = 2,\ w = \dfrac{1}{2} + iv,\ v \in [0.1, 11.7]$

This shows that $\Im\left(\frac{a+c_w a^{2-2w}}{1-2w}\right) \geqslant 0$ when $\Re(w) = \frac{1}{2}$. In fact, it is strictly positive if $\Re(w) > \frac{1}{2}$ (Lax and Phillips, 1976).

Two more pictures of $(a + c_w a^{2-2w})/(1-2w)$



$a = 2,\ w \in \frac{1}{2} + [5, 11.7]\,i$



$a = 2,\ w \in 0.55 + [5, 11.7]\,i$

A necessary condition to be satisfied (Peter Sarnak and Tom Spencer)

Let us write

$$F := \begin{pmatrix} \theta_{\mathcal{D}}(u_{\mathcal{D},w}) & \theta_{\mathcal{D}}(v_{w,a}) \\ \\ \eta_a(u_{\mathcal{D},w}) & \eta_a(v_{w,a}) \end{pmatrix}$$

and F^* for the complex conjugate of the transpose. Then it must be that the matrix $C := (F - F^*)/(2i)$ is a positive definite hermitian matrix, hence $\det C > 0$. Since $\eta(u) = \theta(v)$, this means that the condition

$$\Im(\theta_{\mathcal{D}}(u_{\mathcal{D},w})) \cdot \Im(\eta_a(v_{w,a})) > \Im(\theta_{\mathcal{D}}(v_{w,a}))^2$$

must be pointwise satisfied when $\Re(w) > \frac{1}{2}$.

This condition is stronger than the non-vanishing of the resolvent $\theta(u)\eta(v) - \theta(v)^2$. Compare with the preceding picture arising from a 1×1 matrix, rather than 2×2.

The explicit formula

More explicitly,

$$\Im\left(\frac{1}{4\pi i}\int_{(\frac{1}{2})}|E_s(\mathcal{D})|^2\,\frac{\mathrm{d}s}{\lambda_s-\lambda_w}\right)\times\Im\left(\frac{a+c_w a^{2-2w}}{1-2w}\right)$$
$$\geqslant\left\{\Im\left[\frac{1}{1-2w}\left(a^{1-w}E_w(\mathcal{D})-R_w(\mathcal{D},a)\right)\right]\right\}^2.$$

Recalling that $a=\sqrt{D}/(2K)$, when $K\to\infty$ and $w=u+iv$ is fixed with $\frac{1}{2}<u<1$ and $v\neq 0$, this yields when $\alpha\to 0$ the asymptotic inequality

$$\Im\left(\frac{1}{4\pi i}\int_{(\frac{1}{2})}|E_s(\mathcal{D})|^2\frac{\mathrm{d}s}{\lambda_s-\lambda_w}\right)\times\Im\left(\frac{1}{1-2w}\right)$$
$$\gtrsim 2K\times\left\{\Im\left(\frac{K^{w-1}\zeta(w)}{4(1-2w)(w+2)\zeta(w+2)}\right)\right\}^2 D^{\frac{5}{2}}.$$

Since we can take $D\to\infty$ and $K\to\infty$ (slowly!) and since $w=u+iv$ is at our disposal with $\frac{1}{2}<u<1$, this proves that as $D\to\infty$ the left-hand side is of order strictly greater than $D^{\frac{5}{2}}$.

Open problem

FIND AN ASYMPTOTIC FORMULA FOR

$$\frac{1}{4\pi i}\int_{(\frac{1}{2})}|E_s(\mathcal{D})|^2\frac{\mathrm{d}s}{\lambda_s^2}$$

Remark By a well-known estimate of Jutila, it is not difficult to show that the order of magnitude in question does not exceed $D^{\frac{5}{2}}\log^A D$ for some not too large A, while we have shown in a roundabout way that it cannot be $D^{\frac{5}{2}}$.

THE END