The Riemann-Roch strategy

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(Collaboration with C. Consani)

RH equivalent

RH problem is equivalent to an inequality for real valued functions f on $\mathbb{R}_+^*$ of the form

$$\text{RH} \iff \mathfrak{s}(f,f) \leq 0\,,\ \forall f \mid \int f(u)d^*u = \int f(u)du = 0.$$

$$\mathfrak{s}(f,g) := N(f \star \tilde{g}), \quad \tilde{g}(u) := u^{-1}g(u^{-1})$$

$\star$ = convolution product on $\mathbb{R}_+^*$

$$N(h) := \sum_{n=1}^{\infty} \Lambda(n)h(n) + \int_1^{\infty} \frac{u^2h(u) - h(1)}{u^2 - 1} d^*u + c\, h(1)$$

$$c = \frac{1}{2}(\log \pi + \gamma).$$

Explicit Formula

$F : [1, \infty) \to \mathbb{R}$ continuous and continuously differentiable except for finitely many points at which both $F(u)$ and $F'(u)$ have at most a discontinuity of the first kind, and s.t. for some $\epsilon > 0 : F(u) = O(u^{-1/2-\epsilon})$

$$\Phi(s) = \int_1^\infty F(u)\, u^{s-1} du$$

$$\Phi(\frac{1}{2}) + \Phi(-\frac{1}{2}) - \sum_{\rho \in \text{Zeros}} \Phi(\rho - \frac{1}{2}) = \sum_p \sum_{m=1}^{\infty} \log p\; p^{-m/2} F(p^m) +$$

$$+ (\frac{\gamma}{2} + \frac{\log \pi}{2}) F(1) + \int_1^\infty \frac{t^{3/2} F(t) - F(1)}{t(t^2-1)} dt$$

Weil's formulation

$h \in \mathcal{S}(C_{\mathbb{K}})$ a Schwartz function with compact support :

$$\widehat{h}(0) + \widehat{h}(1) - \sum_{\chi \in \widehat{C_{\mathbb{K},1}}} \sum_{Z_{\tilde{\chi}}} \widehat{h}(\tilde{\chi}, \rho) = \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|} \, d^*u$$

where the principal value $\int'_{\mathbb{K}_v^*}$ is normalized by the additive character α_v and for any character ω of $C_{\mathbb{K}}$

$$\widehat{h}(\omega, z) := \int h(u)\, \omega(u)\, |u|^z \, d^*u, \quad \widehat{h}(t) := \widehat{h}(1, t)$$

The adele class space and the explicit formulas

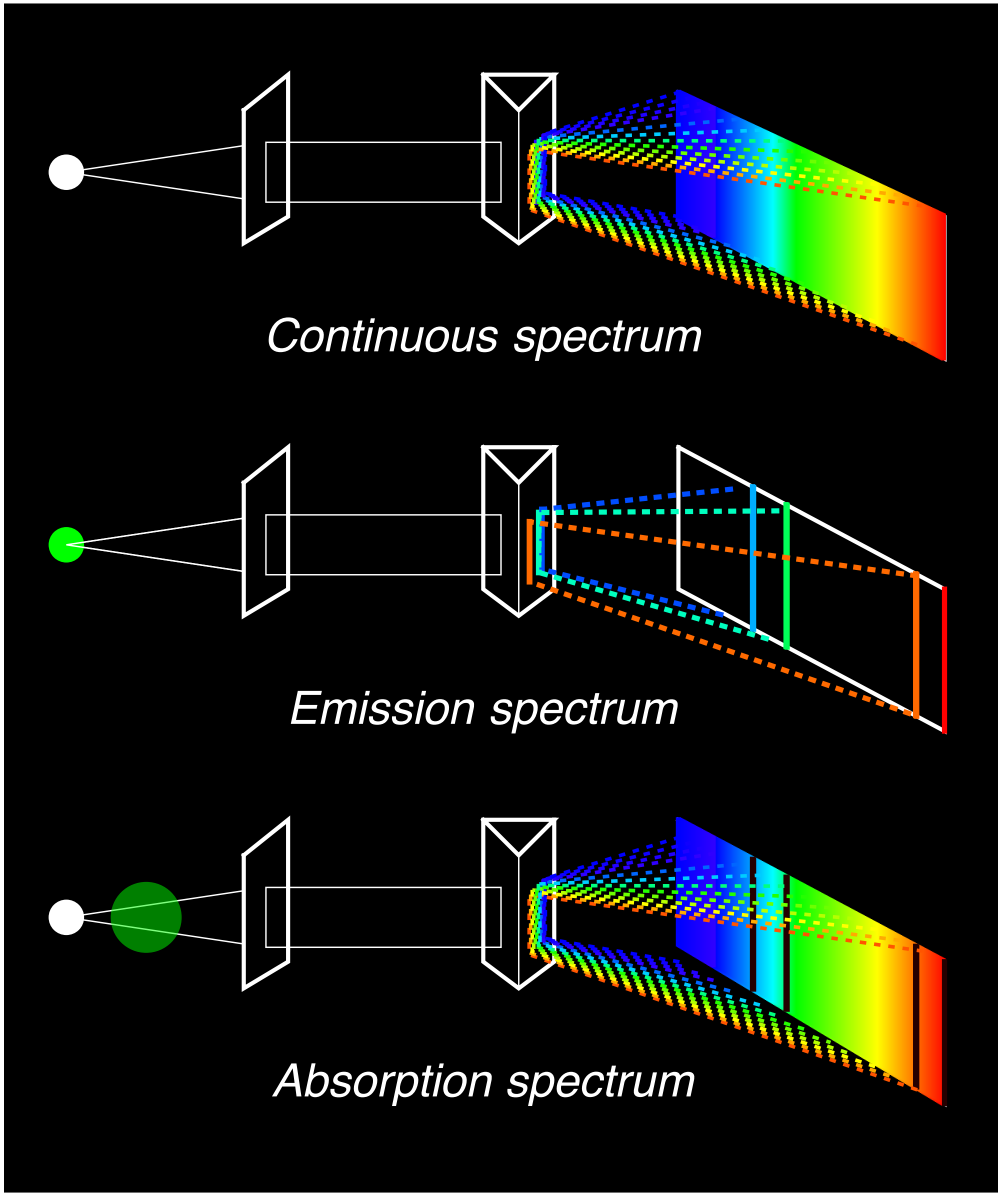
Let $\mathbb{K}$ be a global field, the adele class space of $\mathbb{K}$ is the quotient $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^\times$ of the adeles of $\mathbb{K}$ by the action of $\mathbb{K}^\times$ by multiplication.

$$T\xi(x) := \xi(ux) = \int k(x,y)\xi(y)dy$$

$$k(x,y) = \delta(ux - y),$$

$$\text{Tr}_{\text{distr}}(T) := \int k(x,x)dx = \int \delta(ux - x)dx$$

$$= \frac{1}{|u-1|}\int \delta(z)dz = \frac{1}{|u-1|}$$


Continuous spectrum
Emission spectrum
Absorption spectrum

Critical zeros as absorption spectrum

The spectral side now involves **all** non-trivial zeros and the geometric side is given by :

$$\mathrm{Tr}_{\mathrm{distr}}\left(\int h(w)\vartheta(w)d^*w\right) = \sum_v \int_{\mathbb{K}_v^\times} \frac{h(w^{-1})}{|1-w|}\, d^*w$$

(A. Connes, Selecta 1998, R. Meyer, Duke, 2005)

The limit $q \to 1$ and the Hasse-Weil formula

C. Soulé : $\zeta_X(s) := \lim_{q \to 1} Z(X, q^{-s})(q-1)^{N(1)}, \qquad s \in \mathbb{R}$

where $Z(X, q^{-s})$ denotes the evaluation at $T = q^{-s}$ of the Hasse-Weil exponential series

$$Z(X, T) := \exp\left(\sum_{r \geq 1} N(q^r) \frac{T^r}{r}\right)$$

For the projective space $\mathbb{P}^n$: $\quad N(q) = 1 + q + \ldots + q^n$

$$\zeta_{\mathbb{P}^n(\mathbb{F}_1)}(s) = \lim_{q \to 1} (q-1)^{n+1} \zeta_{\mathbb{P}^n(\mathbb{F}_q)}(s) = \frac{1}{\prod_0^n (s-k)}$$

The limit $q \to 1$

The Riemann sums of an integral appear from the right hand side :

$$\frac{\partial_s \zeta_N(s)}{\zeta_N(s)} = -\int_1^\infty N(u)\, u^{-s} d^*u$$

Thus the integral equation produces a precise equation for the **counting function** $N_C(q) = N(q)$ associated to the hypothetical curve C :

$$\frac{\partial_s \zeta_{\mathbb{Q}}(s)}{\zeta_{\mathbb{Q}}(s)} = -\int_1^\infty N(u)\, u^{-s} d^*u$$

The distribution $N(u)$

This equation admits a solution which is a **distribution** and is given with $\varphi(u) := \sum_{n<u} n\,\Lambda(n)$, by the equality

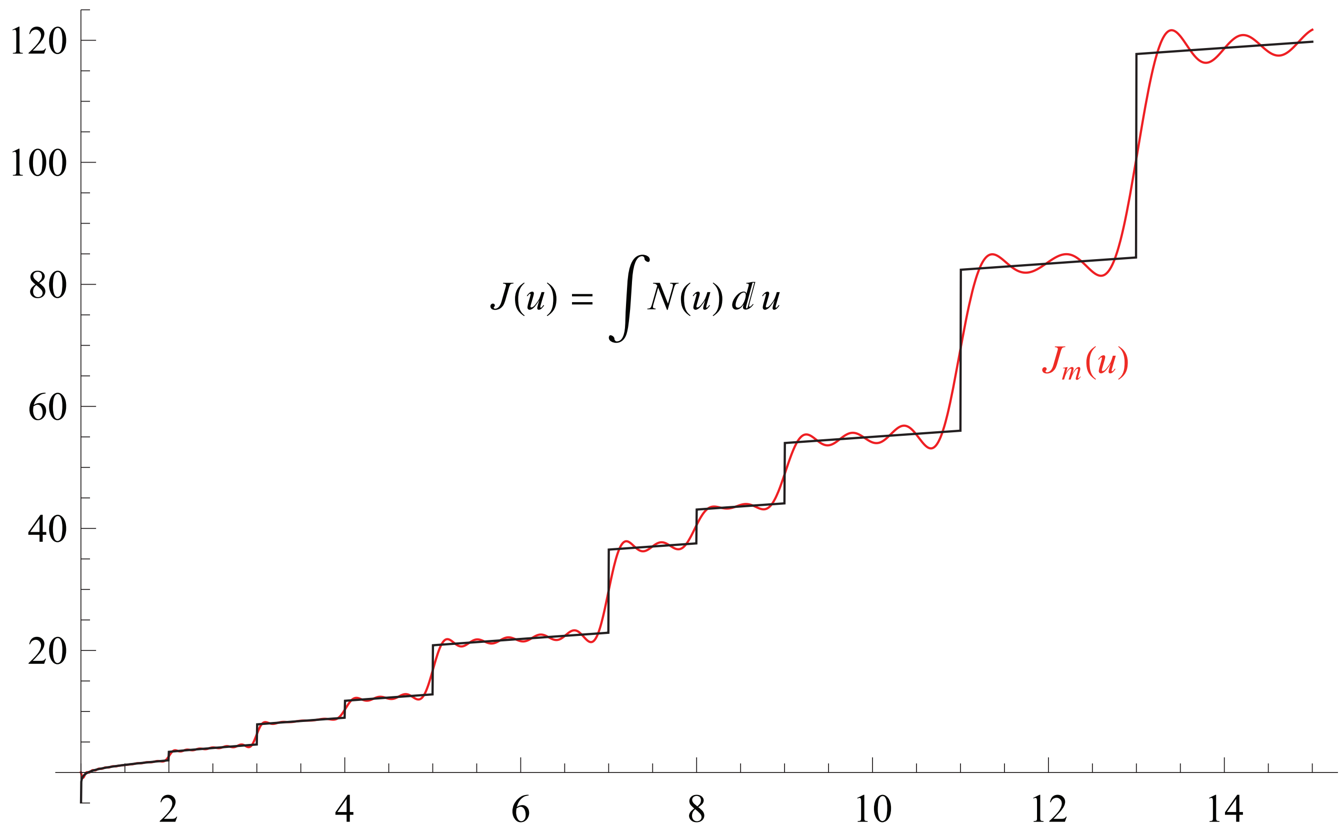
$$N(u) = \frac{d}{du}\varphi(u) + \kappa(u)$$

where $\kappa(u)$ is the distribution which appears in the explicit formula

$$\int_1^\infty \kappa(u) f(u) d^*u = \int_1^\infty \frac{u^2 f(u) - f(1)}{u^2 - 1} d^*u + c f(1)\,, \quad c = \frac{1}{2}(\log \pi + \gamma)$$

The conclusion is that the distribution $N(u)$ is **positive** on $(1, \infty)$ and is given by

$$N(u) = u - \frac{d}{du}\left(\sum_{\rho \in Z} \text{order}(\rho) \frac{u^{\rho+1}}{\rho+1}\right) + 1$$


120
100
80
60
40
20
2
4
6
8
10
12
14
$J(u) = \int N(u)\, du$
$J_m(u)$

Space $X_{\mathbb{Q}} := \mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^{\times}$

The quotient $X_{\mathbb{Q}} := \mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}/\hat{\mathbb{Z}}^{\times}$ of the adele class space $\mathbb{Q}^{\times}\backslash\mathbb{A}_{\mathbb{Q}}$ of the rational numbers by the maximal compact subgroup $\hat{\mathbb{Z}}^{\times}$ of the idele class group, gives by considering the induced action of $\mathbb{R}_{+}^{\times}$, the above counting distribution $N(u)$, $u \in [1, \infty)$, which determines, using the Hasse-Weil formula in the limit $q \to 1$, the complete Riemann zeta function.

Geometric structure of $X_{\mathbb{Q}}$

The action of $\mathbb{R}_+^\times$ on the space $X_{\mathbb{Q}}$ is in fact the action of the Frobenius automorphisms Fr_λ on the points of the arithmetic site over $\mathbb{R}_+^{\max}$.

Topos + characteristic 1

— Arithmetic Site.

— Frobenius correspondences.

— Extension of scalars to $\mathbb{R}_+^{\max}$.

Why semirings ?

A category $\mathcal{C}$ is *semiadditive* if it has finite products and corpoducts, the morphism $0 \to 1$ is an isomorphism (thus $\mathcal{C}$ has a 0), and the morphisms

$$\gamma_{M,N} : M \vee N \to M \times N$$

are isomorphisms.
Then End(M) is naturally a semiring for any object M.

Finite semifields, characteristic 1

$\mathbb{K} =$ finite semifield : then $\mathbb{K}$ is a field or $\mathbb{K} = \mathbb{B}$:

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

The semifield $\mathbb{Z}_{\text{max}}$

Lemma : Let F be a semifield of characteristic 1, then for $n \in \mathbb{N}^{\times}$ the map $\text{Fr}_n \in \text{End}(\mathbb{F})$, $\text{Fr}_n(x) := x^n \ \forall x \in F$ defines an **injective endomorphism** of F.

$\mathbb{Z}_{\text{max}} := (\mathbb{Z} \cup \{-\infty\}, \max, +)$, unique semifield with multiplicative group infinite cyclic.
multiplicative notation : Addition $\vee$, $u^n \vee u^m = u^k$, *with* $k = \sup(n, m)$. *Multiplication :* $u^n u^m = u^{n+m}$.

Map $\mathbb{N}^{\times} \to \text{End}(\mathbb{Z}_{\text{max}})$, $n \mapsto \text{Fr}_n$ is isomorphism of semigroups. (extend to 0)

Arithmetic Site $(\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$

$\mathbb{Z}_{\max}$ on which $\mathbb{N}^{\times}$ acts by $n \mapsto \mathrm{Fr}_n$ is a semiring in the topos $\widehat{\mathbb{N}^{\times}}$ of sets with an action of $\mathbb{N}^{\times}$.

The *Arithmetic Site* $(\widehat{\mathbb{N}^{\times}}, \mathbb{Z}_{\max})$ is the topos $\widehat{\mathbb{N}^{\times}}$ endowed with the *structure sheaf* : $\mathcal{O} := \mathbb{Z}_{\max}$ semiring in the topos.

Characteristic 1

The role of $\mathbb{F}_q$ is played by

$$\mathbb{B} := \{0, 1\}, \quad 1 + 1 = 1$$

No finite extension, but

$\mathrm{Fr}_\lambda(x) = x^\lambda$ automorphisms of $\mathbb{R}_+^{\max}$.

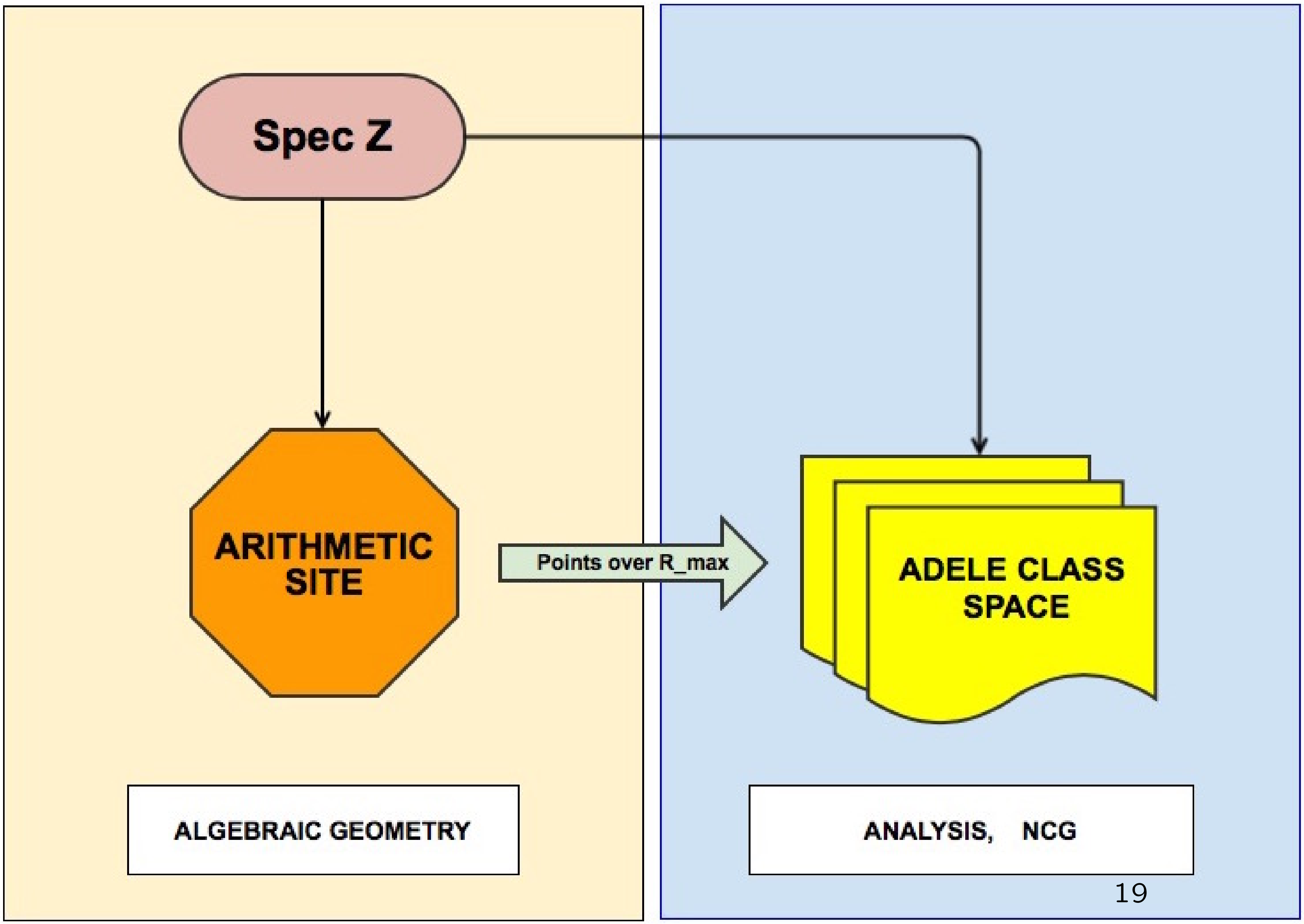
$$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max}) = \mathbb{R}_+^\times$$

Points of the arithmetic site over $\mathbb{R}_+^{\max}$

These are defined as pairs $(p, f_p^\#)$ of a point p of $\widehat{\mathbb{N}^\times}$ and local morphism $f_p^\# : \mathcal{O}_p \to \mathbb{R}_+^{\max}$.

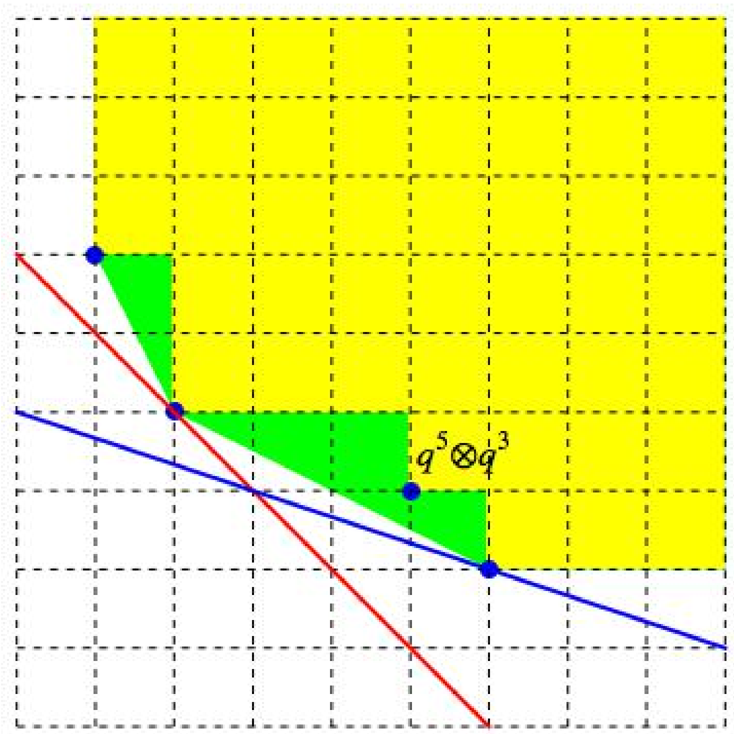
Theorem

The points $\mathcal{A}(\mathbb{R}_+^{\max})$ of $(\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ on $\mathbb{R}_+^{\max}$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \hat{\mathbb{Z}}^*$. The action of the Frobenius Fr_λ of $\mathbb{R}_+^{\max}$ corresponds to the action of the idele class group.


Spec Z
ARITHMETIC SITE
Points over R_max
ADELE CLASS SPACE
ALGEBRAIC GEOMETRY
ANALYSIS, NCG

C curve defined over $\mathbb{F}_q$	**Arithmetic Site** $\mathcal{A} = (\widehat{\mathbb{N}^\times}, \mathbb{Z}_{\max})$ over $\mathbb{B}$
Structure sheaf $\mathcal{O}_C$	**Structure sheaf** $\mathbb{Z}_{\max}$
Galois on $C(\bar{\mathbb{F}}_q)$	$\mathrm{Gal}_{\mathbb{B}}(\mathbb{R}_+^{\max})$ **on** $\mathcal{A}(\mathbb{R}_+^{\max})$
Ψ Frobenius Correspondence	**Correspondences** $\Psi(\lambda)$ $\lambda \in \mathbb{R}_+^*$ **on** $\mathcal{A} \times \mathcal{A}$

Frobenius Correspondences



Theorem

Let $\lambda, \lambda' \in \mathbb{R}_+^*$ with $\lambda\lambda' \notin \mathbb{Q}$. The composition

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda')$$

Same if λ and λ' are rational. If $\lambda \notin \mathbb{Q}, \lambda' \notin \mathbb{Q}$ and $\lambda\lambda' \in \mathbb{Q}$,

$$\Psi(\lambda) \circ \Psi(\lambda') = \Psi(\lambda\lambda') \circ \mathrm{Id}_\epsilon = \mathrm{Id}_\epsilon \circ \Psi(\lambda\lambda')$$

where Id_ϵ is the tangential deformation of Id.

Divisors and intersection

Intersection $D \bullet D'$ of formal divisors

$$D := \int h(\lambda)\Psi_\lambda \, d^*\lambda$$

$$D \bullet D' := < D \star \tilde{D}', \Delta >$$

where $\tilde{D}'$ is the transposed D' and composition $D \star \tilde{D}'$ is bilinear $< D \star \tilde{D}', \Delta >$ using the distribution $N(u)$ and correspondence Ψ_λ of degree λ.

Negativity $\iff$ RH

- Horizontal and vertical ξ_j.

- RH is equivalent to inequality

$$D \bullet D \leq 2(D \bullet \xi_1)(D \bullet \xi_2)$$

Incompatibility of $\leq$ with naive positivity resolved by small lemma (cf Matuck-Tate-Grothendieck)

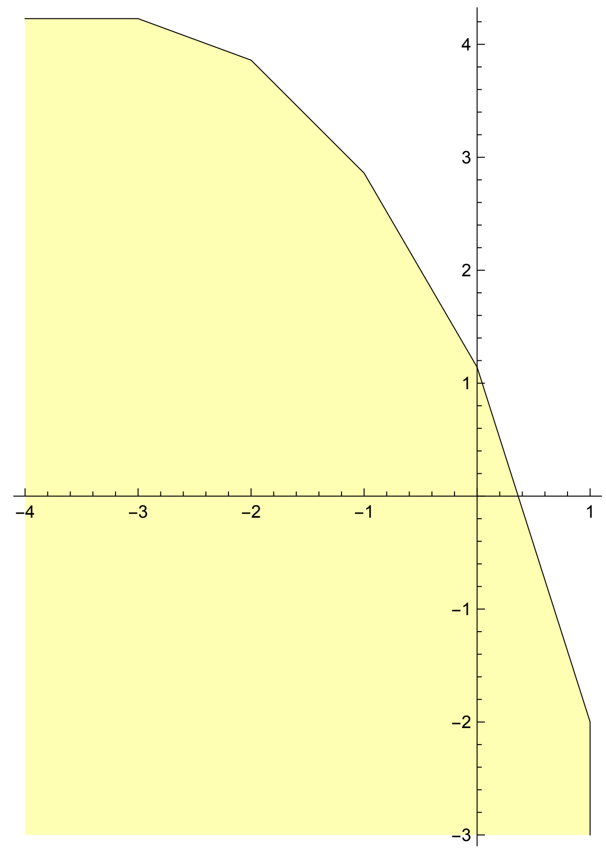
Extension of scalars to $\mathbb{R}_{\max}$

The following holds :

$$\mathbb{Z}_{\max} \widehat{\otimes}_{\mathbb{B}} \mathbb{R}_{\max} \simeq \mathcal{R}(\mathbb{Z})$$

$\mathcal{R}(\mathbb{Z})$ = semiring of continuous, convex, piecewise affine functions on $\mathbb{R}_+$ with slopes in $\mathbb{Z} \subset \mathbb{R}$ and only finitely many discontinuities of the derivative

These functions are endowed with the pointwise operations of functions with values in $\mathbb{R}_{\max}$



4
3
2
1
−1
−2
−3
−4
−3
−2
−1
1

Points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$

Theorem : The points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ form the double quotient $\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q} / \widehat{\mathbb{Z}}^*$.

Corollary : One has a canonical isomorphism Θ between the points of the topos $[0, \infty) \rtimes \mathbb{N}^\times$ and $\mathcal{A}(\mathbb{R}_+^{\max})$ *i.e.* the points of the arithmetic site defined over $\mathbb{R}_+^{\max}$.

Structure sheaf of $[0,\infty) \rtimes \mathbb{N}^{\times}$

This is the sheaf on $[0,\infty) \rtimes \mathbb{N}^{\times}$ associated to convex, piecewise affine functions with integral slopes

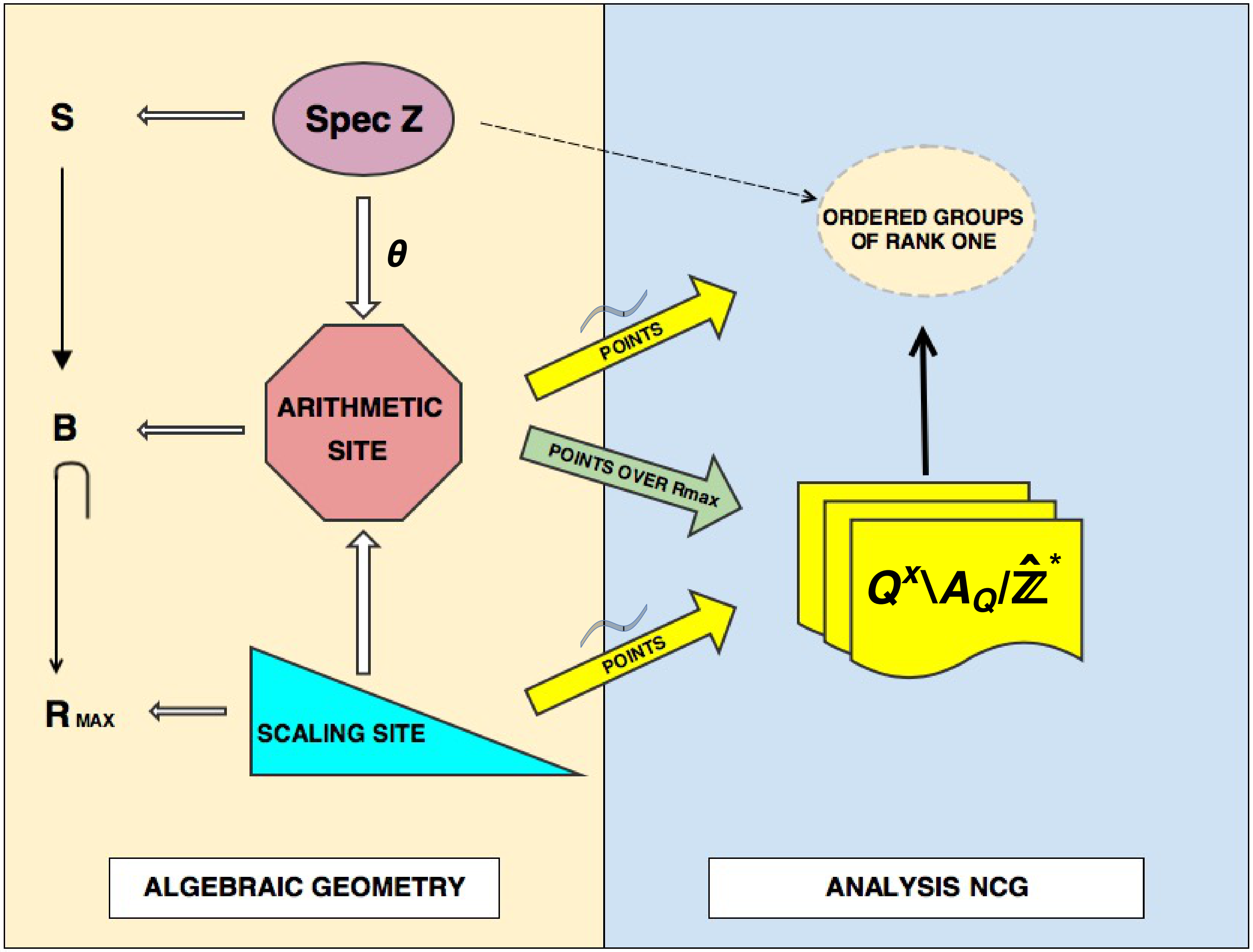
Same as for the localization of zeros of analytic functions $f(X) = \sum a_n X^n$ in an annulus

$$A(r_1, r_2) = \{z \in K \mid r_1 < |z| < r_2\}$$

$$\tau(f)(x) := \max_n \{-nx - v(a_n)\}, \ \forall x \in (-\log r_2, -\log r_1)$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta$$

$\bar{C} = C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$ on $\bar{\mathbb{F}}_q$	**Scaling site** $\hat{\mathcal{A}} = ([0,\infty) \rtimes \mathbb{N}^\times, \mathcal{O})$ on $\mathbb{R}_+^{\max}$
$C(\bar{\mathbb{F}}_q) = \bar{C}(\bar{\mathbb{F}}_q)$	$\mathcal{A}(\mathbb{R}_+^{\max}) = \hat{\mathcal{A}}(\mathbb{R}_+^{\max})$
Structure sheaf $\mathcal{O}_{\bar{C}}$ of $\bar{C}$ $= C \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$	$\mathbb{Z}_{\max} \hat{\otimes}_{\mathbb{B}} \mathbb{R}_+^{\max} \to$ Sheaf of **convex piecewise affine functions, slopes** $\in \mathbb{Z}$
Sheaf $\mathcal{K}$ of rational functions	Sheaf of fractions $=$ **continuous piecewise affine functions, slopes** $\in \mathbb{Z}$


S
Spec Z
θ
ORDERED GROUPS
OF RANK ONE
POINTS
ARITHMETIC
SITE
B
POINTS OVER Rmax
$Q^{x}\backslash A_Q/\hat{\mathbb{Z}}^*$
POINTS
R MAX
SCALING SITE
ALGEBRAIC GEOMETRY
ANALYSIS NCG

Periodic Orbits

By restriction of the structure sheaf of

$$\hat{\mathcal{A}} = ([0,\infty) \rtimes \mathbb{N}^{\times}, \mathcal{O})$$

to periodic orbits (*i.e.* **the image of** $\mathrm{Spec}\,\mathbb{Z}$**) one obtains, for each prime** p **a real analogue**

$$C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$$

of Jacobi elliptic curve $\mathbb{C}^*/q^{\mathbb{Z}}$**.**

Elliptic curve over $\mathbb{C}$	**Periodic orbit** Curve C_p over $\mathbb{R}_+^{\max}$
Points over $\mathbb{C}$: $\mathbb{C}^\times/q^{\mathbb{Z}}$	$\mathbb{R}_+^*/p^{\mathbb{Z}}$, $H \subset \mathbb{R}$, $H \sim H_p$
Structure sheaf periodic functions $f(qz) = f(z)$	Sheaf of periodic convex piecewise affine functions, **slopes** $\in H_p$
Sheaf $\mathcal{K}$ of rational functions $f(qz) = f(z)$	Sheaf of periodic $f(p\lambda) = f(\lambda)$ continuous piecewise affine functions, **slopes** $\in H_p$

Rational functions

For $W \subset C_p$ open, $\mathcal{O}_p(W)$ is simplifiable, one lets $\mathcal{K}_p$ the sheaf associated to the presheaf $W \mapsto \text{Frac}\,\mathcal{O}_p(W)$.

Lemma The sections of the sheaf $\mathcal{K}_p$ are continuous piecewise affine functions with slopes in H_p endowed with max ($\vee$) and the sum.

$$(x - y) \vee (z - t) = ((x + t) \vee (y + z)) - (y + t).$$

Cartier divisors

Lemma : The sheaf CDiv(C_p) of Cartier divisors *i.e.* the quotient sheaf $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, is isomorphic to the sheaf of naive divisors $H \mapsto D(H) \in H$,

$$\forall \lambda, \ \exists V \text{ open } \lambda \in V, \ D(\mu) = 0\,, \ \forall \mu \in V, \ \mu \neq \lambda$$

Point $\mathfrak{p}_H$ associated to $H \subset \mathbb{R}$ and f section of $\mathcal{K}$ at $\mathfrak{p}_H$.

$$\text{Order}(f) = h_+ - h_- \in H \subset \mathbb{R}$$

$$h_\pm = \lim_{\epsilon \to 0\pm} \frac{f((1+\epsilon)H) - f(H)}{\epsilon}$$

.

Divisors

Definition : A divisor is a global section of $\mathcal{K}_p^\times/\mathcal{O}_p^\times$, *i.e.* a map $H \to D(H) \in H$ vanishing except on finitely many points.

Proposition : (i) The divisors $\text{Div}(C_p)$ form an abelian group under addition.
(ii) The condition $D'(H) \geq D(H)$, $\forall H \in C_p$, defines a partial order on $\text{Div}(C_p)$.
(iii) The **degree** map is additive and order preserving

$$\deg(D) := \sum D(H) \in \mathbb{R}.$$

Principal divisors

The sheaf $\mathcal{K}_p$ admits global sections :

$$\mathcal{K} := \mathcal{K}(C_p) = H^0(\mathbb{R}_+^*/p^{\mathbb{Z}}, \mathcal{K}_p)$$

the semifield of global sections.

Principal divisors : The map which to $f \in \mathcal{K}^\times$ associates the divisor

$$(f) := \sum_H (H, \mathrm{Ord}_H(f)) \in \mathrm{Div}(C_p)$$

is a group morphism $\mathcal{K}^\times \to \mathcal{P} \subset \mathrm{Div}(C_p)$.

The subgroup $\mathcal{P} \subset \mathrm{Div}(C_p)$ of principal divisors is **contained in the kernel** of the morphism $\deg : \mathrm{Div}(C_p) \to \mathbb{R}$:

$$\sum_H \mathrm{Ord}_H(f) = 0\,,\ \forall f \in \mathcal{K}^\times.$$

Invariant χ

For $p > 2$ one considers the ideal $(p-1)H_p \subset H_p$.

$$0 \to (p-1)H_p \to H_p \xrightarrow{r} \mathbb{Z}/(p-1)\mathbb{Z} \to 0$$

Lemma : For $H \subset \mathbb{R}$, $H \simeq H_p$, the map $\chi : H \to \mathbb{Z}/(p-1)\mathbb{Z}$, $\chi(\mu) = r(\mu/\lambda)$ where $H = \lambda H_p$ is independent of the choice of λ.

Theorem

The map $(\deg, \chi)$ is a **group isomorphism**

$$(\deg, \chi) : \mathrm{Div}(C_p)/\mathcal{P} \to \mathbb{R} \times (\mathbb{Z}/(p-1)\mathbb{Z})$$

where $\mathcal{P}$ is the subgroup of principal divisors.

Theta Functions on $C_p = \mathbb{R}_+^*/p^{\mathbb{Z}}$

$$\prod_0^\infty (1 - t^m w) \to f_+(\lambda) := \sum_0^\infty \left(0 \vee (1 - p^m \lambda)\right)$$

$$\prod_1^\infty (1 - t^m w^{-1}) \to f_-(\lambda) := \sum_1^\infty \Big(0 \vee (p^{-m}\lambda - 1)\Big)$$

Theorem

Any $f \in \mathcal{K}(C_p)$ has a canonical decomposition

$$f(\lambda) = \sum_i \Theta_{h_i,\mu_i}(\lambda) - \sum_j \Theta_{h'_j,\mu'_j}(\lambda) - h\lambda + c$$

where $c \in \mathbb{R}$, $(p-1)h = \sum h_i - \sum h'_j$ and $h_i \le \mu_i < p h_i$, $h'_j \le \mu_j < p h'_j$.

p-adic filtration $H^0(D)^\rho$

Definition : Let $D \in \mathrm{Div}(C_p)$ one lets

$$H^0(D) := \{f \in \mathcal{K}(C_p) \mid D + (f) \geq 0\}$$

It is an $\mathbb{R}_{\max}$-module, $f, g \in H^0(D) \Rightarrow f \vee g \in H^0(D)$.

Lemma : Let $D \in \mathrm{Div}(C_p)$ be a divisor, one gets a filtration of $H^0(D)$ by $\mathbb{R}_{\max}$-sub-modules :

$$H^0(D)^\rho := \{f \in H^0(D) \mid \|f\|_p \leq \rho\}$$

using the p-adic norm.

Real valued Dimension

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) := \lim_{n\to\infty} p^{-n}\mathrm{dim}_{\mathrm{top}}(H^0(D)^{p^n})$$

where the *topological dimension* $\mathrm{dim}_{\mathrm{top}}(X)$ is the number of real parameters on which solutions depend.

Riemann-Roch Theorem

(i) Let $D \in \mathrm{Div}(C_p)$ a divisor with $\deg(D) \geq 0$, then

$$\lim_{n\to\infty} p^{-n}\mathrm{dim}_{\mathrm{top}}(H^0(D)^{p^n}) = \deg(D)$$

(ii) One has the Riemann-Roch formula :

$$\mathrm{Dim}_{\mathbb{R}}(H^0(D)) - \mathrm{Dim}_{\mathbb{R}}(H^0(-D)) = \deg(D)\,,\ \forall D \in \mathrm{Div}(C_p).$$

Back to the goal : RR on the square

Integrals of Frobenius correspondences

$$D := \int h(\lambda)\Psi_\lambda \, d^*\lambda$$

One needs a Riemann-Roch formula

$$\dim H^0 - \dim H^1 + \dim H^2 = \frac{1}{2} D \bullet D$$

in order to make $\pm D$ effective and get a contradiction (Negativity $\iff$ RH)

Open problem : suitable definition of H^1

Tropical RR theorem

Baker, Norine, Gathmann, Kerber.

The power in these results is the existence part, it uses

Game theory, Potential theory

but the definition of the terms in the RR formula are **not** given in terms of the dimension of H^0 !
(counter-example of Yoshitomi)

Complex lift of the Scaling Site

The new development in our strategy is to deduce the existence part of the Riemann-Roch formula in the tropical shadow (*i.e.* on the square of the Scaling Site) from a corresponding formula holding on the analytic geometric version of the space (i.e. its complex lift)

The advantage of working in characteristic zero is to have already available all the algebraic and analytical tools needed to prove such result

Jensen

$f(z)$ holomorphic function in an annulus

$$A(r_1, r_2) = \{z \in \mathbb{C} \mid r_1 < |z| < r_2\}$$

$$\tau(f)(x) := \frac{1}{2\pi} \int_0^{2\pi} \log |f(e^{-x+i\theta})| d\theta.$$

$$\exists z \mid f(z) = 0, \; -\log|z| = x \iff \Delta(\tau(f))(x) \neq 0$$

Tropical zeros of $\tau(f)$ are the $-\log|z|$.

Tropical descent

$$D + (\tau) := \sum n_j \delta_{\lambda_j} + \Delta(\tau) \geq 0.$$

First attempt : punctured disk

$\mathbb{D}^* \rtimes \mathbb{N}^\times \to [0, \infty) \rtimes \mathbb{N}^\times$

$$\mathbb{D}^* := \{q \in \mathbb{C} \mid 0 < |q| \leq 1\}$$

The monoid $\mathbb{N}^\times$ acts naturally on $\mathbb{D}^*$ by means of the map $q \mapsto q^n$. In this way, one defines a ringed topos by endowing the topos $\mathbb{D}^* \rtimes \mathbb{N}^\times$ with the structure sheaf $\mathcal{O}$ of complex analytic functions.

The map

$$\mathbb{D}^* \ni q \mapsto -\log|q| \in [0, \infty)$$

extends to a **geometric morphism** of toposes $\mathbb{D}^* \rtimes \mathbb{N}^\times \to [0, \infty) \rtimes \mathbb{N}^\times$.

Almost periodic analytic fcts

In order to lift divisors of the form $D(f) = \int f(\lambda)\delta_\lambda \, d^*\lambda$ to a **discrete** divisor $\tilde{D}(f)$ on a complex geometric space, one uses the **Jessen theory** of analytic almost periodic functions

$$\varphi(\sigma) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \log|f(\sigma + it)|dt \qquad (1)$$

$$\lim_{T\to\infty} \frac{N(T)}{2T} = \frac{\varphi'(\sigma_2) - \varphi'(\sigma_1)}{2\pi}. \qquad (2)$$

Proetale cover $\tilde{\mathbb{D}}^* \rtimes \mathbb{N}^{\times}$

$$\tilde{\mathbb{D}}^* := \varprojlim_{\mathbb{N}^{\times}} (\mathbb{D}^*, z \mapsto z^n).$$

One uses : Witt construction in characteristic 1 and Teichmuller lift $[-] : \mathbb{R}_+^{\max} \to W$, to define

$$q(z) := [|z|] \exp(2\pi i \arg z)$$

The structure sheaf of the pro-étale cover involves the **ring** $W[q^r]$ generated by rational powers q^r of q over W

Compare to perfectoid torus.

Adelic description : $\mathcal{C}_{\mathbb{Q}}$

Compactification $G := \varprojlim_{\mathbb{N}^\times} \mathbb{R}/n\mathbb{Z}$,

$$\mathcal{C}_{\mathbb{Q}} = \mathbb{Q}^* \backslash (\mathbb{A}_{\mathbb{Q}} \times G) = P(\mathbb{Q}) \backslash \overline{P(\mathbb{A}_{\mathbb{Q}})}. \qquad (3)$$

The obtained noncommutative space is the moduli space of elliptic curves endowed with a triangular structure, up to isogenies.

A **triangular structure** on an elliptic curve E is a pair (ξ, η) of elements of the Tate module $T(E)$, such that $\xi \neq 0$ and $< \xi^\perp, \eta >= \mathbb{Z}$.

$\xi^\perp := \{\chi \in \mathrm{Hom}(E, \mathbb{R}/\mathbb{Z}) \mid T(\chi)(\xi) = 0\} \subset \mathrm{Hom}(E, \mathbb{R}/\mathbb{Z})$

The space $\mathcal{C}_{\mathbb{Q}}$ has a foliation ! of complex dimension 1 and an additional real deformation parameter.

Frobenius correspondences

Use the Witt construction in characteristic 1, entropy

$$u + v = \sup_{\alpha \in [0,1]} c(\alpha) u^\alpha v^{1-\alpha}, \ c(\alpha) := \alpha^{-\alpha}(1-\alpha)^{1-\alpha}$$

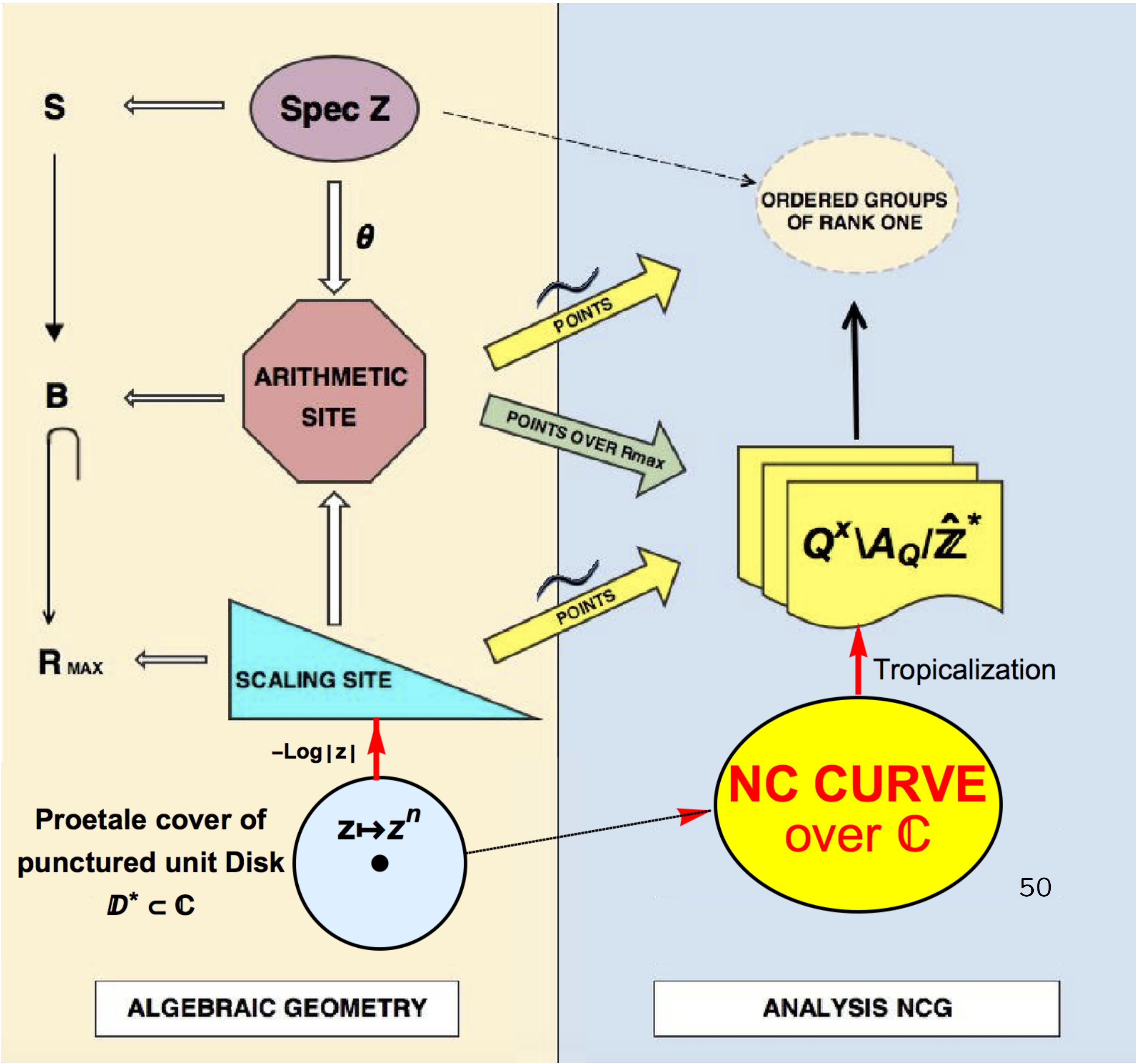
Automorphisms $\theta_\lambda \in \mathrm{Aut}(W)$, Teichmüller lift $[x]$

$$\theta_\lambda([x]) = [x^\lambda], \ \forall x \in \mathbb{R}_+^{\max}, \ \lambda \in \mathbb{R}_+^*.$$

The right action $R(\mu)$ of $\mathbb{R}_+^* \subset P_+(\mathbb{R})$ extends to W valued functions, the arithmetic Frobenius is

$$f \mapsto \mathrm{Fr}_\mu^a(f), \ \mathrm{Fr}_\mu^a(f) := \theta_\mu(R(\mu^{-1})(f))$$

$$q(x+iy) := [e^{-2\pi y}] e^{2\pi i x}$$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$z \mapsto z^{n}$
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ALGEBRAIC GEOMETRY
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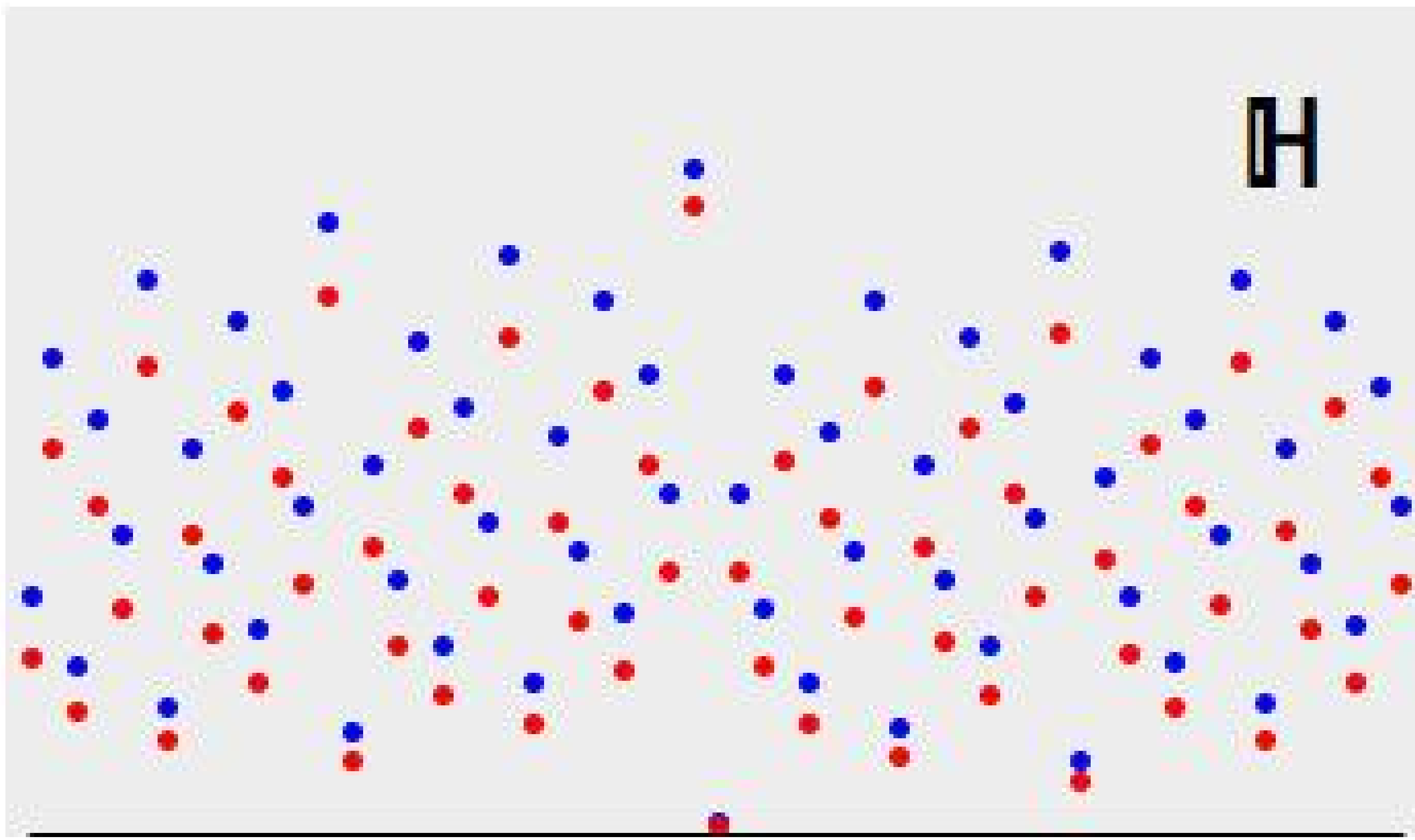
Strategy

1. Develop intersection theory in such a way that the divergent term in $\log \Lambda$ is eliminated.
2. Formulate and prove a Hirzebruch-Riemann-Roch formula on the square whose topological side part $\frac{1}{2}c_1(E)^2$ is $\frac{1}{2}\mathfrak{s}(f,f)$. This step involves the lifting $D(f) = \int f(\lambda)\Psi_\lambda \, d^*\lambda$ to a divisor $\tilde{D}(f)$ in the complex set-up and the use of correspondences.
3. Use the assumed positivity of $\mathfrak{s}(f,f)$ to get an existence result for $H^0(\tilde{D}(f))$ or $H^0(-\tilde{D}(f))$.
4. Use tropical descent to get the effectivity of a divisor equivalent to $D(f)$ and finally get a contradiction.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