L-functions and Random Matrix Theory

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and
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June 6, 2018

In 1972 Hugh Montgomery - during a conversation with Freeman Dyson - discovered that the zeros of the Riemann zeta-function are distributed like the eigenvalues of random GUE matrices.

H. L. MONTGOMERY

Sarvadaman Chowla introduced Montgomery to Dyson at tea at IAS

Freeman Dyson

Montgomery, 1971 – pair correlation

$$\sum_{\gamma_1,\gamma_2\in[0,T]} w(\gamma_1-\gamma_2) f\left(\frac{\log T}{2\pi}(\gamma_1-\gamma_2)\right)$$
$$= \frac{T\log T}{2\pi}\left(f(0)+\int_{-\infty}^{\infty} f(u)\left[1-\left(\frac{\sin(\pi u)}{\pi u}\right)^2\right]du + o(1)\right)$$

where the Fourier transform of f vanishes outside of $[-1,1]$ and $w(x)=4/(4+x^2)$.

Note from Dyson to Selberg with a reference to Mehta's book.

THE INSTITUTE FOR ADVANCED STUDY
PRINCETON, NEW JERSEY 08540

SCHOOL OF NATURAL SCIENCES

April 7 1972

Dear Atle

The reference which Dr Montgomery wants is

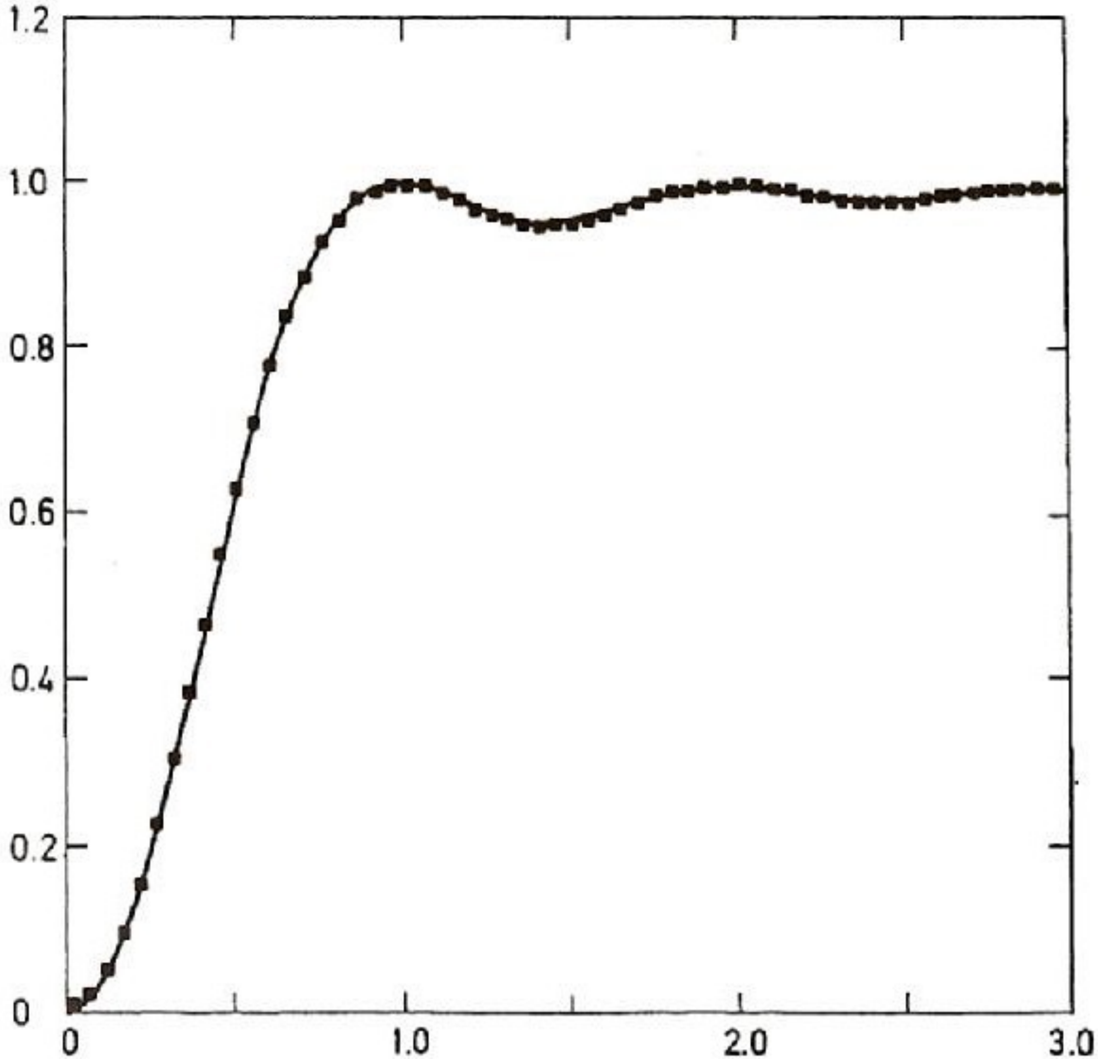
M. L. Mehta, "Random Matrices"
Academic Press, N.Y. 1967.

Page 76 Equation 6.13

Page 113 Equation 9.61

showing that the pair-correlation function of zeros of the ζ-function is identical ~~which~~ with that of eigenvalues of a random complex (Hermitian or unitary) matrix of large order.

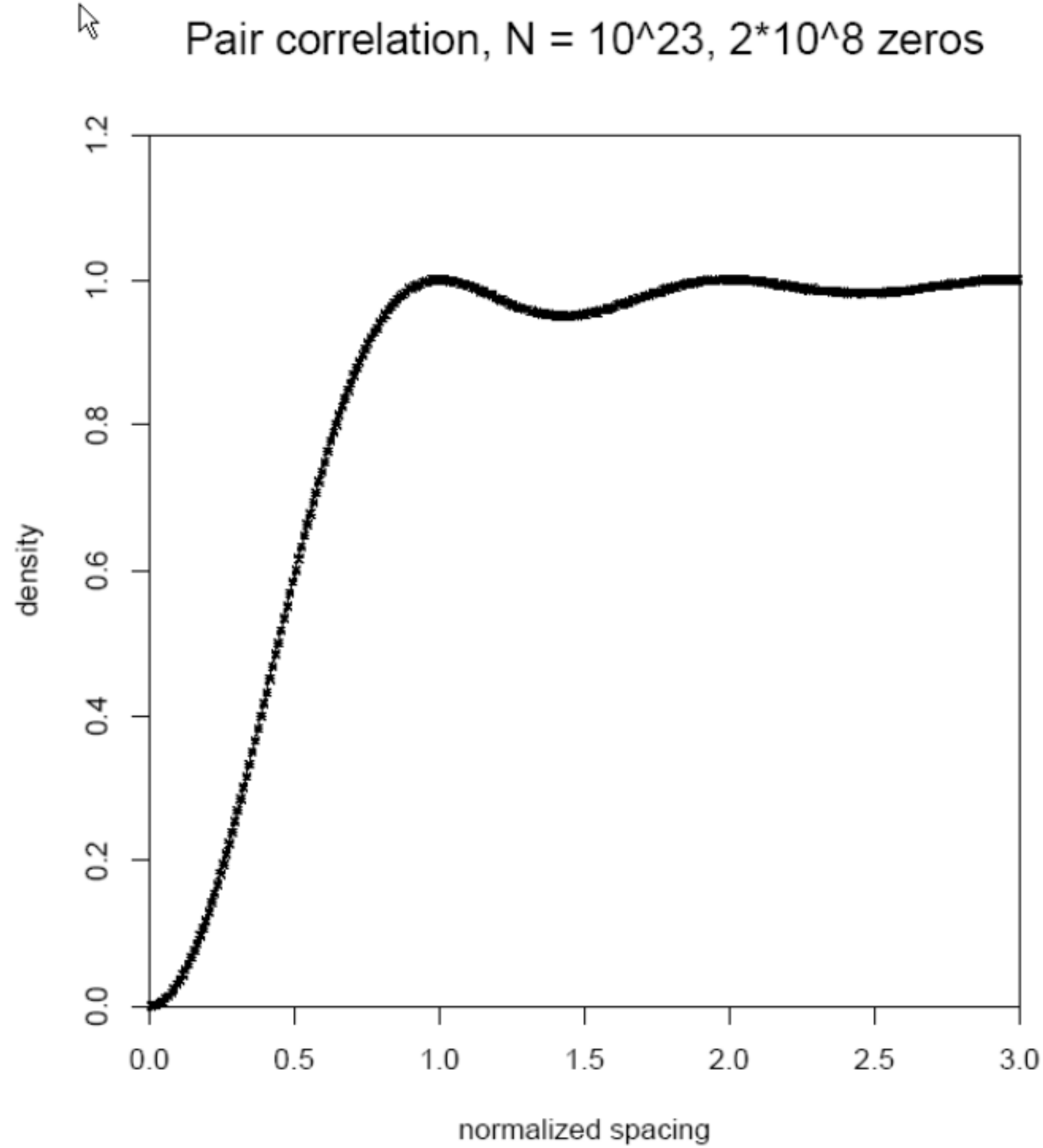
Freeman Dyson.



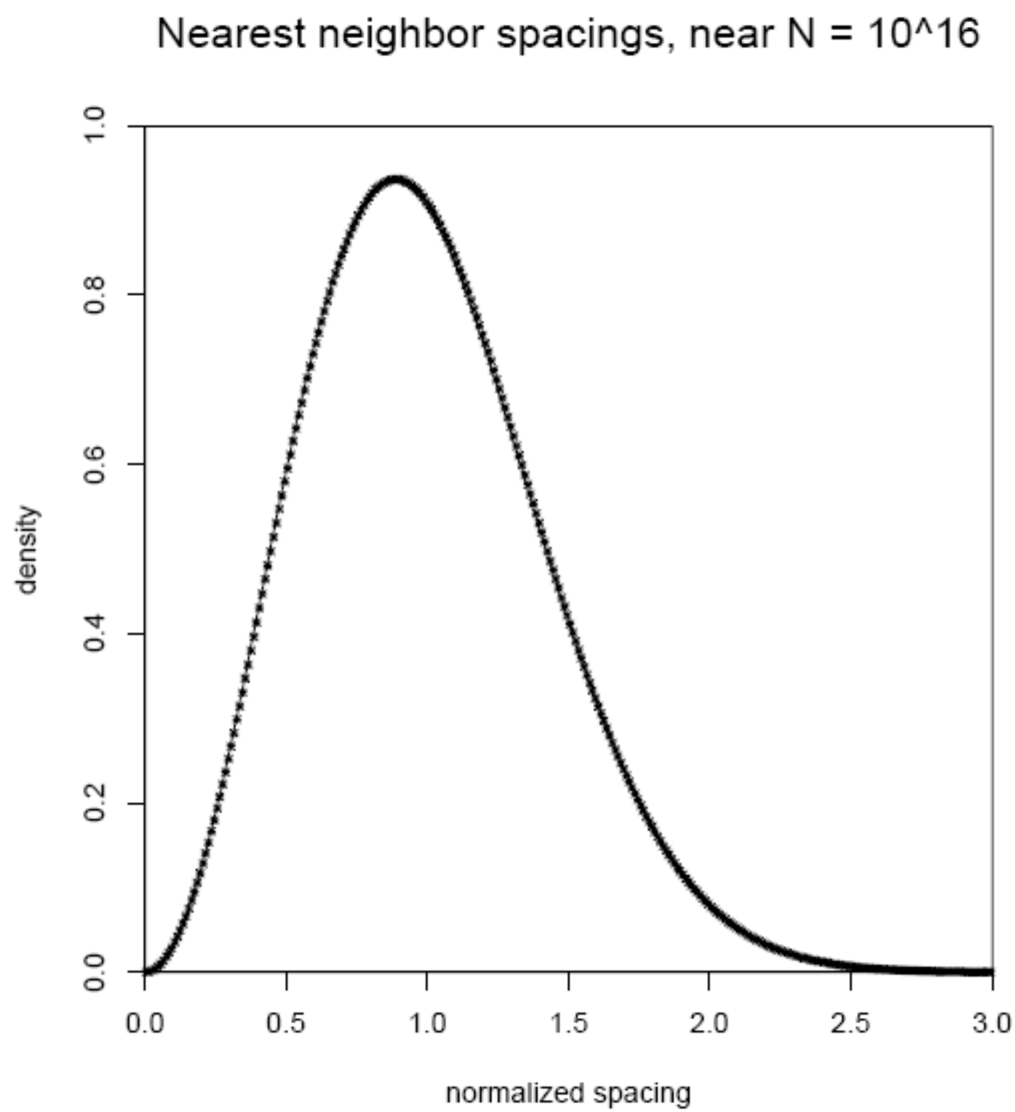
Picture by
A. Odlyzko

79 million zeros
around the
10^{20} th zero

$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$



Pair correlation, N = 10^23, 2*10^8 zeros
density
normalized spacing
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0.0
0.5
1.0
1.5
2.0
2.5
3.0



Nearest neighbor spacings, near N = 10^16
density
normalized spacing
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Bohigas, Giannoni, Schmit, Berry, & Tabor were the first to understand the RMT implications of data on nuclear levels.

Sir Michael Berry

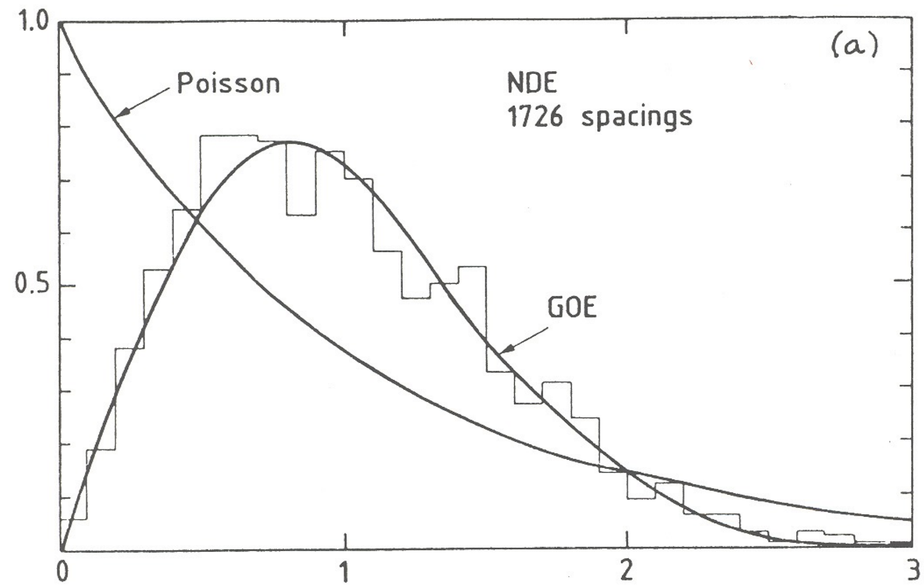
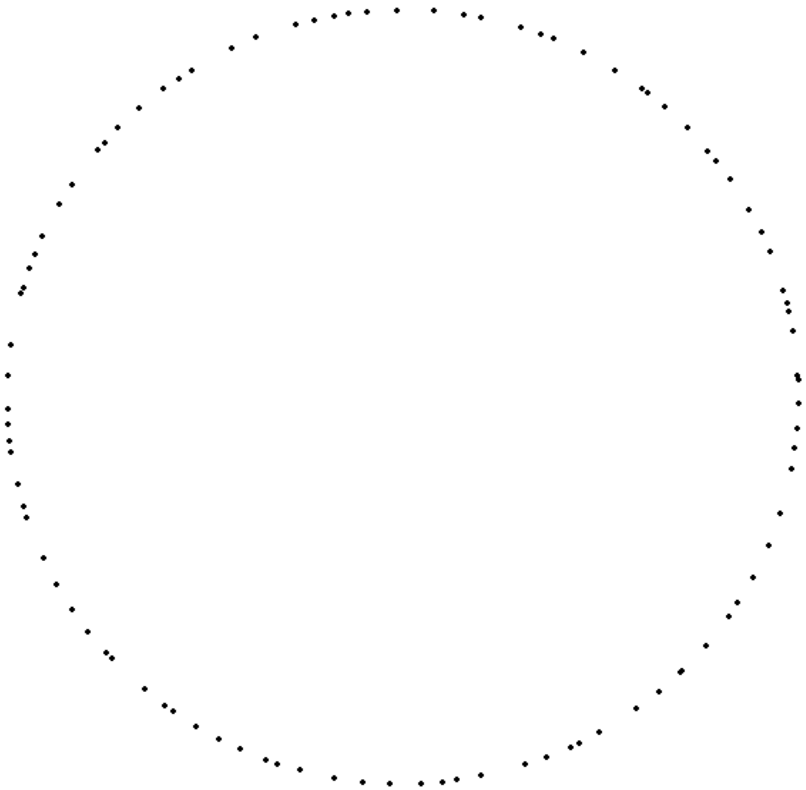
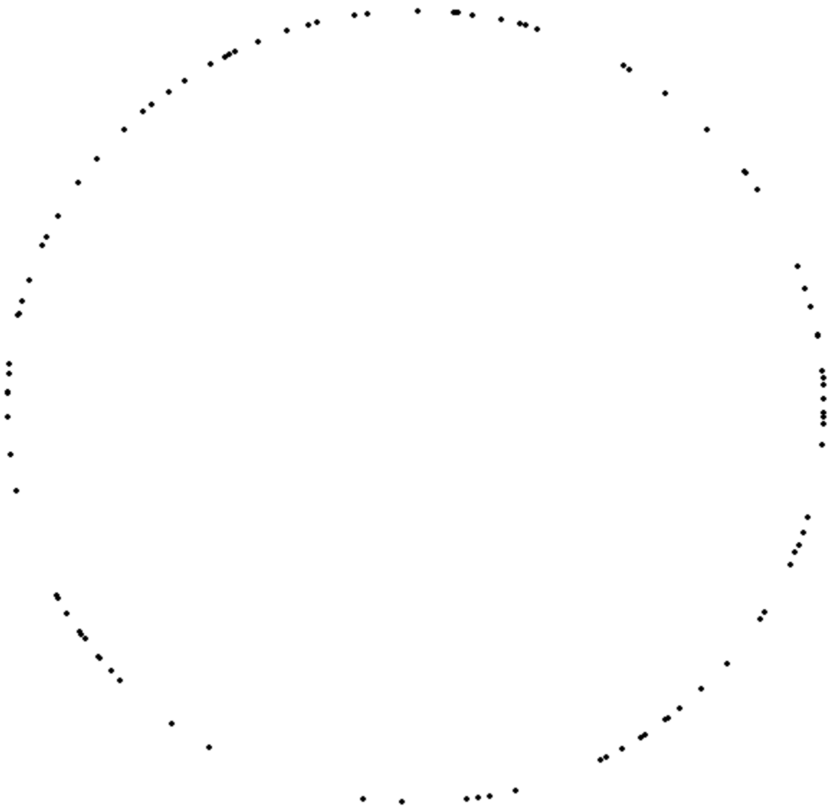


FIG. 1.4. Level spacing histogram for a large set of nuclear levels, often referred to as nuclear data ensemble. The data considered consist of 1407 resonance levels belonging to 30 sequences of 27 different nuclei: (i) slow neutron resonances of Cd(110, 112, 114), Sm(152, 154), Gd(154, 156, 158, 160), Dy(160, 162, 164), Er(166, 168, 170), Yb(172, 174, 176), W(182, 184, 186), Th(232), and U(238) (1146 levels); (ii) proton resonances of Ca(44) (J=1/2+), Ca(44) (J=1/2−), and Ti(48) (J=1/2+) (157 levels); and (iii) (n,γ)-reaction data on Hf(177) (J=3), Hf(177) (J=4), Hf(179) (J=4), and Hf(179) (J=5) (104 levels). The data chosen in each sequence is believed to be complete (no missing levels) and pure (the same angular momentum and parity).

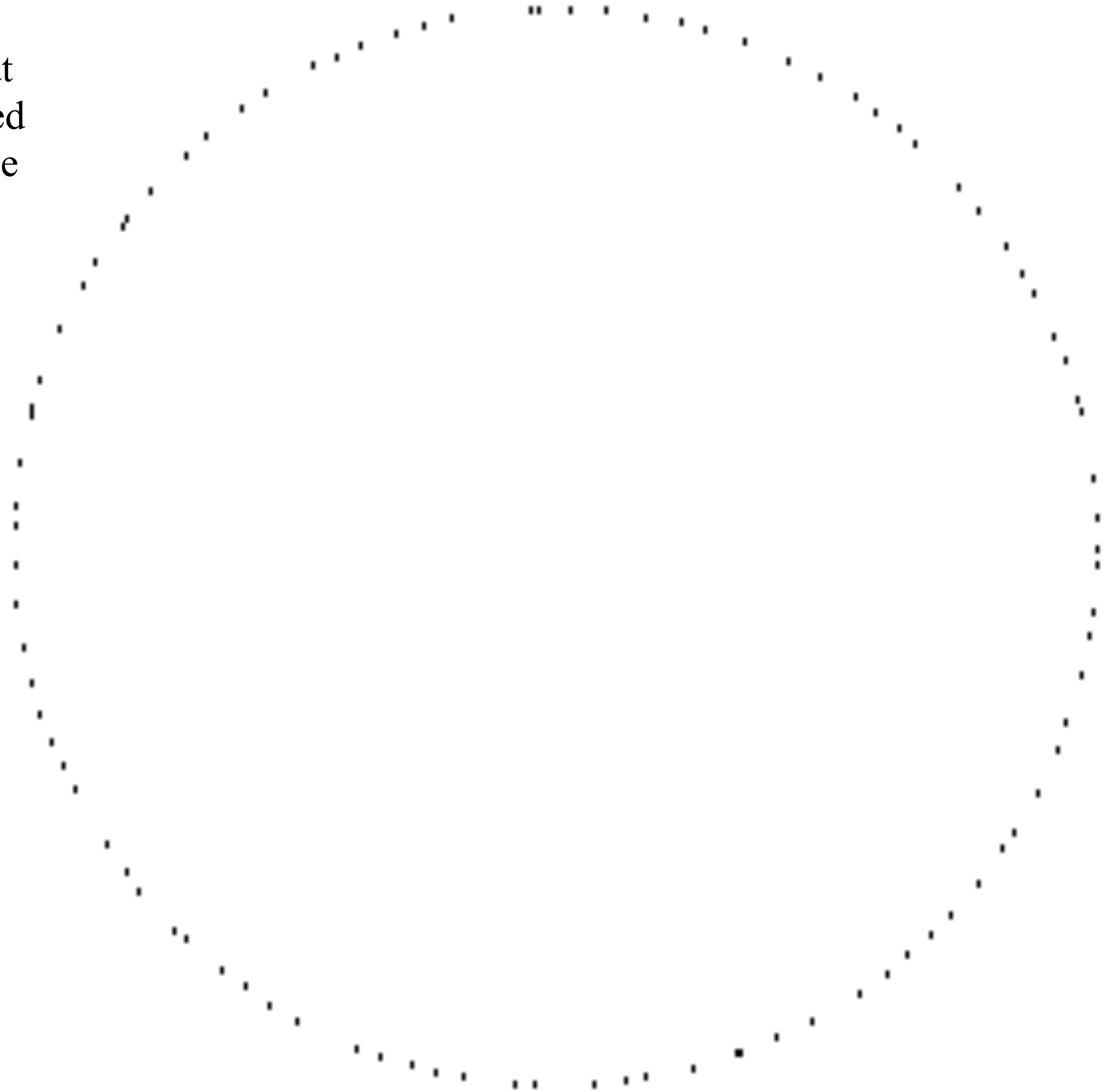




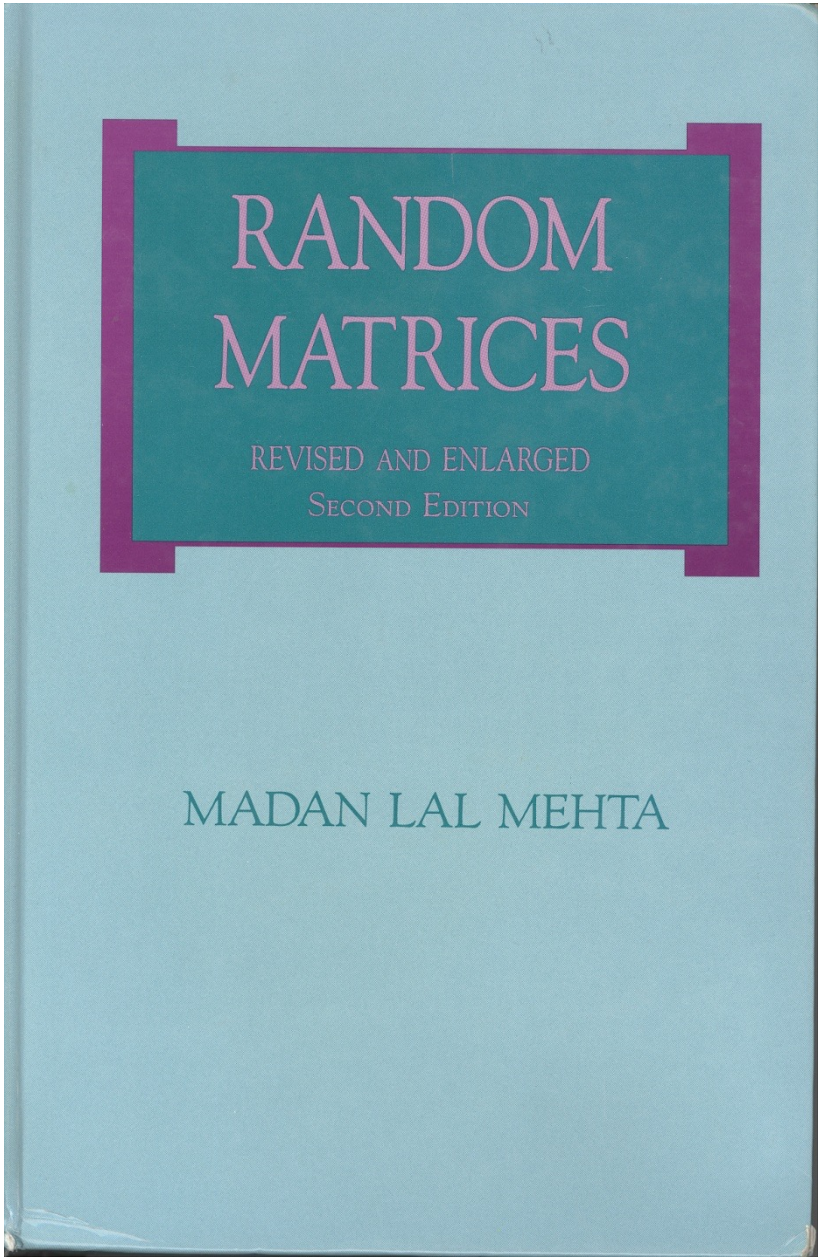
Eigenvalues of a randomly generated 96 X 96 unitary matrix

96 points chosen at random on [0,1] and mapped to the circle by exp(2 pi I x)

96 zeros of zeta starting at a height 1200 and wrapped once around the unit circle



Mehta's book is the classic reference on random matrix theory.



Berry's number variance calculations (1988)

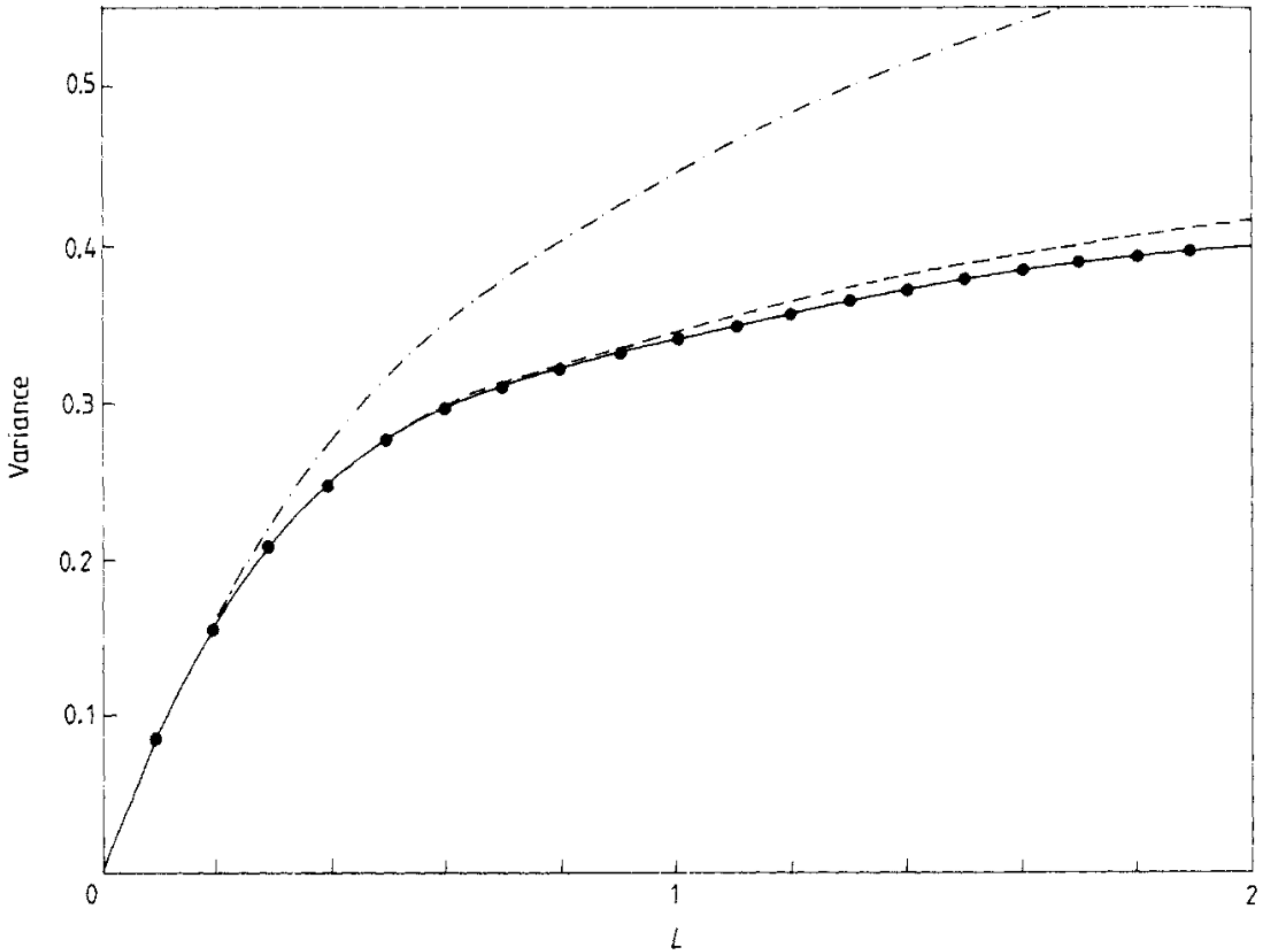


Figure 1. Number variance $V(L;x)$ of the Riemann zeros, for $0 \leqslant L \leqslant 2$ and $x = 10^{12}$. Dots: computed from the zeros by Odlyzko; full curve: semiclassical formula (19) with $\tau^* = \frac{1}{4}$; broken curve: number variance of the GUE; chain curve: number variance of the Gaussian orthogonal ensemble (GOE) of real symmetric random matrices.

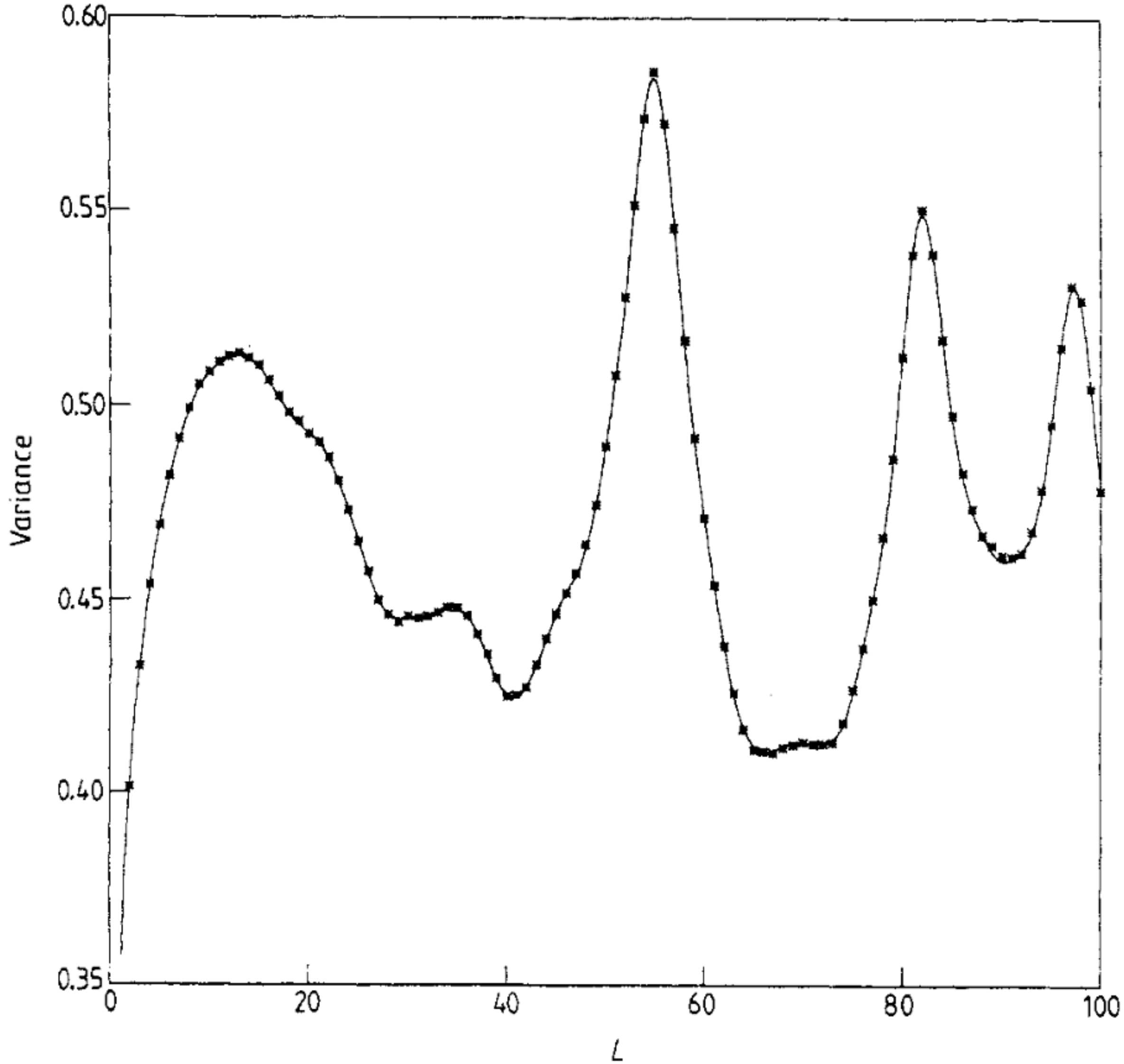


Figure 3. As figure 2 but for $0 \leqslant L \leqslant 100$ and with stars rather than dots for the variance computed from the zeros (GUE not shown).

Ozluk (1982): “Pair Correlation of Zeros of Dirichlet L-functions”

Ozluk-Snyder (1993): “Small zeros of quadratic *L*-functions"

Hejhal (1994): “On the triple correlation of zeros of the zeta function”

Rudnick and Sarnak (1995): “Zeros of principal L-functions and random matrix theory”

Symmetry and families of L-functions

Katz and Sarnak (1999) discovered that the classical compact groups come into play when one considers families of L-functions over function fields; in particular the symplectic and orthogonal groups.

Nick Katz

Peter Sarnak

This is suggestive of a spectral interpretation of zeros of L-functions.

Iwaniec, Luo, and Sarnak (2000) “Low lying zeros of L-functions” gave ample theoretical evidence for the ideas of Katz and Sarnak for families of L-functions over number fields.

Moments

Classical Mean-Value theorems for ζ

Hardy and Littlewood (1918):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 \, dt \sim \log T$$

Ingham (1926):

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt \quad \sim \quad 2\prod_p (1-\frac{1}{p})(1+\frac{1}{p})\frac{\log^4 T}{4!}$$

Conrey and Ghosh conjecture: mid 1980's

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^{2k}\ dt \sim g_k a_k \frac{\log^{k^2} T}{k^2!}$$

where

$$a_k = \prod_p \left(1-\frac{1}{p}\right)^{(k-1)^2} \times \left(1+\frac{\binom{k-1}{1}^2}{p}+\frac{\binom{k-1}{2}^2}{p^2}+\dots\right)$$

and g_k is an integer.

Conrey and Ghosh conjecture: 1992

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^6\,dt \sim 42\prod_p \left(1-\frac{1}{p}\right)^4\left(1+\frac{4}{p}+\frac{1}{p^2}\right)\frac{\log^9 T}{9!}$$

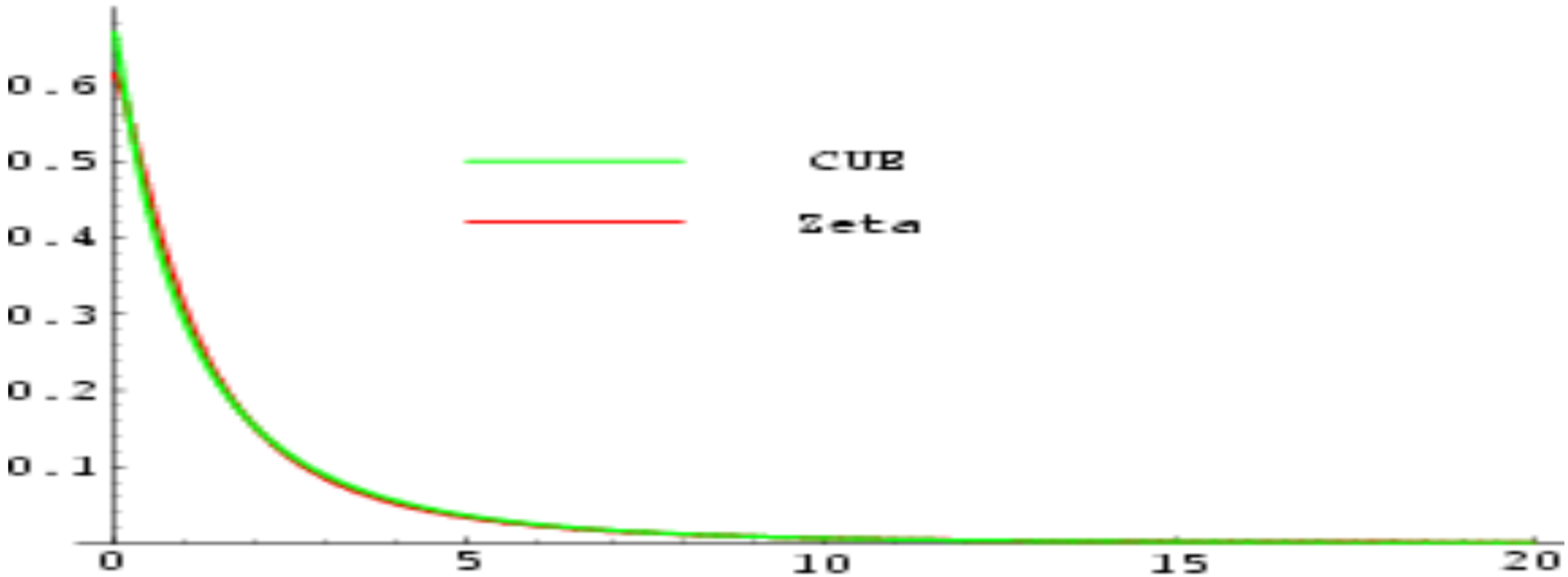
Conrey and Gonek conjecture: 1998

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^8\,dt \sim 24024\prod_p \left(1-\frac{1}{p}\right)^9\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)\frac{\log^{16} T}{16!}$$

In 1998, Jon Keating and Nina Snaith discovered a close connection between the distribution of values of zeta and the distribution of values of characteristic polynomials of unitary matrices.

Jon Keating, Bristol University

Nina Snaith, Bristol University



Distribution of values of zeta vs RMT (Nina Snaith)

Keating and Snaith formula:

$$\int_{U(N)} |\det(I-X)|^{2k} dX = \frac{(N+1)(N+2)^2\ldots(N+k)^k(N+k+1)^{k-1}\ldots(N+2k+1)}{1\cdot 2^2\cdot k^k\cdot(k+1)^{k-1}\cdot(2k+1)}$$

$$\sim g_k \frac{N^{k^2}}{k^2!}$$

$$g_k = \frac{k^2!}{1\cdot 2^2\cdot k^k\cdot(k+1)^{k-1}\cdot(2k+1)}$$

$$g_1 = 1 \qquad g_2 = 2 \qquad g_3 = 42 \qquad g_4 = 24024$$

$$g_5 = 701149020$$

More precise moments: Lower order terms

Ingham:

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^2 \, dt = \log\frac{T}{2\pi} + 2\gamma - 1 + O(T^{1/2+\epsilon})$$

Heath-Brown (1979)

$$\frac{1}{T}\int_0^T |\zeta(1/2+it)|^4 \, dt = P(\log T) + O(T^{7/8+\epsilon})$$

for some polynomial P of degree 4

Theorem:

$$\int_0^T |\zeta(1/2+it)|^4 \ dt$$
$$= \int_0^T P_2\left(\log \frac{t}{2\pi}\right) \ dt + O(T^{2/3+\epsilon})$$

where

$$\begin{aligned} P_2(x) \ = \ & \frac{1}{2\pi^2}x^4 + \frac{8}{\pi^4}\left(\gamma\pi^2 - 3\zeta'(2)\right)x^3 \\ & + \frac{6}{\pi^6}\left(-48\gamma\zeta'(2)\pi^2 - 12\zeta''(2)\pi^2 + 7\gamma^2\pi^4 + 144\zeta'(2)^2 - 2\,\gamma_1\pi^4\right)x^2 \\ & + \frac{12}{\pi^8}(6\gamma^3\pi^6 - 84\gamma^2\zeta'(2)\pi^4 + 24\gamma_1\zeta'(2)\pi^4 - 1728\zeta'(2)^3 + 576\gamma\zeta'(2)^2\pi^2 \\ & \qquad + 288\zeta'(2)\zeta''(2)\pi^2 - 8\zeta'''(2)\pi^4 - 10\gamma_1\gamma\pi^6 - \gamma_2\pi^6 - 48\gamma\zeta''(2)\pi^4)x \\ & + \frac{4}{\pi^{10}}(-12\zeta''''(2)\pi^6 + 36\gamma_2\zeta'(2)\pi^6 + 9\gamma^4\pi^8 + 21\gamma_1^2\pi^8 + 432\zeta''(2)^2\pi^4 \\ & \qquad + 3456\gamma\zeta'(2)\zeta''(2)\pi^4 + 3024\gamma^2\zeta'(2)^2\pi^4 - 36\gamma^2\gamma_1\pi^8 - 252\gamma^2\zeta''(2)\pi^6 \\ & \qquad + 3\gamma\gamma_2\pi^8 + 72\gamma_1\zeta''(2)\pi^6 + 360\gamma_1\gamma\zeta'(2)\pi^6 - 216\gamma^3\zeta'(2)\pi^6 \\ & \qquad - 864\gamma_1\zeta'(2)^2\pi^4 + 5\gamma_3\pi^8 + 576\zeta'(2)\zeta'''(2)\pi^4 - 20736\gamma\zeta'(2)^3\pi^2 \\ & \qquad - 15552\zeta''(2)\zeta'(2)^2\pi^2 - 96\gamma\zeta'''(2)\pi^6 + 62208\zeta'(2)^4), \end{aligned}$$

Another look at Ingham:

$$\int_0^T \zeta(s+\alpha)\zeta(1-s+\beta)\ dt = \int_0^T \left(\zeta(1+\alpha+\beta)+e^{-\ell(\alpha+\beta)}\zeta(1-\alpha-\beta)\right)\ dt$$
$$+O(T^{1/2+\epsilon})$$

$\ell = \log \frac{t}{2\pi}$.

Random Matrix analogue

Let $X \in U(N)$ have eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_n}$
and its characteristic polynomial

$$\Lambda_X(s) = \prod_{n=1}^{N}(1 - se^{-i\theta_n}).$$

Let X^* be the conjugate transpose of X. Then

$$\int_{U(N)} \Lambda_X(e^{-\alpha})\Lambda_{X^*}(e^{-\beta})\ dX = z(\alpha+\beta) + e^{-N(\alpha+\beta)}z(-\alpha-\beta)$$

with

$$z(x) = \frac{1}{1-e^{-x}}.$$

$$\int_0^T \zeta(s+\alpha)\zeta(s+\beta)\zeta(1-s+\gamma)\zeta(1-s+\delta)\ dt$$
$$= \int_0^T W(t,\alpha,\beta;\gamma,\delta)\ dt + O(T^{2/3+\epsilon})$$

where $s = 1/2 + it$ and

$$\begin{aligned} W \;=\; & Z(\alpha,\beta;\gamma,\delta) + e^{-\ell(\alpha+\gamma)}Z(-\gamma,\beta;-\alpha,\delta) + e^{-\ell(\alpha+\delta)}Z(-\delta,\beta;\gamma,-\alpha) \\ & + e^{-\ell(\beta+\gamma)}Z(\alpha,-\gamma;-\beta,\delta) + e^{-\ell(\beta+\delta)}Z(\alpha,-\delta;\gamma,-\beta) \\ & + e^{-\ell(\alpha+\beta+\gamma+\delta)}Z(-\gamma,-\delta;-\alpha,-\beta) \end{aligned}$$

and

$$Z(\alpha,\beta;\gamma,\delta) = \frac{\zeta(1+\alpha+\gamma)\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)\zeta(1+\beta+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}.$$

Random Matrix analogue for the fourth moment

$$\int_{U(N)} \Lambda_X(e^{-\alpha})\Lambda_X(e^{-\beta})\Lambda_{X^*}(e^{-\gamma})\Lambda_{X^*}(e^{-\delta})\ dX = W(\alpha,\beta;\gamma,\delta)$$

where

$$\begin{aligned} W \;=\; & Z(\alpha,\beta;\gamma,\delta) + e^{-N(\alpha+\gamma)}Z(-\gamma,\beta;-\alpha,\delta) + e^{-N(\alpha+\delta)}Z(-\delta,\beta;\gamma,-\alpha) \\ & + e^{-N(\beta+\gamma)}Z(\alpha,-\gamma;-\beta,\delta) + e^{-N(\beta+\delta)}Z(\alpha,-\delta;\gamma,-\beta) \\ & + e^{-N(\alpha+\beta+\gamma+\delta)}Z(-\gamma,-\delta;-\alpha,-\beta) \end{aligned}$$

and

$$Z(\alpha,\beta;\gamma,\delta) = z(\alpha+\gamma)z(\alpha+\delta)z(\beta+\gamma)z(\beta+\delta).$$

Theorem (CFKRS). Let

$$Z(A;B) = \prod_{\alpha\in A, \beta\in B} z(\alpha+\beta).$$

Then

$$\int_{U(N)} \prod_{\alpha\in A} \Lambda_X(e^{-\alpha})\Lambda_{X^*}(e^{-\beta})\ dX$$
$$= \sum_{\substack{S\subset A\\ T\subset B\\ |S|=|T|}} e^{-N(\sum s+\sum t)} Z(\overline{S}\cup(-T); \overline{T}\cup(-S)).$$

where

$$\overline{S} = A - S \qquad -S = \{-s : s\in S\} \qquad \sum s = \sum_{s\in S} s$$

Conjecture (CFKRS)

Let $$\prod_{\alpha\in A}\zeta(s+\alpha)=\sum_{n=1}^{\infty}\frac{\tau_A(n)}{n^s}$$

Then (with s=1/2+it)

$$\int_0^T \prod_{\alpha\in A}\zeta(s+\alpha)\prod_{\beta\in B}\zeta(1-s+\beta)\ dt = \int_0^T \sum_{\substack{U\subset A, V\subset B\\ |U|=|V|}} \left(\frac{t}{2\pi}\right)^{-U-V} \mathcal{B}_{A-U+V^-,B-V+U^-}(1)\ dt + O(T^{1-\delta})$$

where $$\mathcal{B}_{A,B}(s)=\sum_{n=1}^{\infty}\frac{\tau_A(m)\tau_B(m)}{m^s}$$

Conjecture (C, Farmer, Keating, Rubinstein, Snaith)

$$\int_0^T |\zeta(1/2+it)|^6 \, dt$$
$$= \int_0^T P_3(\log \frac{t}{2\pi}) \, dt + O(T^{1/2+\epsilon})$$

where

$$\begin{aligned} P_3(x) \;=\;& 0.000000570852703465278839837684144525231\,3x^9 \\ &+ 0.00040502133088411440331215332025984\,x^8 \\ &+ 0.011072455215246998350410400826667\,x^7 \\ &+ 0.14840073080150272680851401518774\,x^6 \\ &+ 1.0459251779054883439385323798059\,x^5 \\ &+ 3.9843850948235347247479640734 29\,x^4 \\ &+ 8.6073191457812067561483476362 9\,x^3 \\ &+ 10.274330830703446134183009522\,x^2 \\ &+ 6.5939130206497581046571339 2\,x \\ &+ 0.91651550763789305901785 43. \end{aligned}$$

$$\int_0^{2350000} |\zeta(1/2+it)|^6 \; dt$$
$$= 3317496016044.9$$

whereas

$$\int_0^{2350000} P3\left(\log\frac{t}{2\pi}\right) \; dt$$
$$= 3317437762612.4$$

Notice that

$$\int_0^{2350000} 42a_3\left(\log\frac{t}{2\pi}\right)^9 \; \frac{dt}{9!}$$
$$= 72925964550.05$$

$$3317496016044.9/72925964550.05 = 45.49$$

Other Moments

Quadratic Dirichlet L-functions



Diaconu, Goldfeld, Hoffstein

First moment: Jutila; Goldfeld and Hoffstein

Second moment: Jutila 

Third moment: Soundararajan; Diaconu, Goldfield, Hoffstein;
Zhang; Diaconu & Whitehead: with an extra main term!

Over function fields: Hoffstein and Rosen; Bucur and Diaconu; Florea; Diaconu

Bucur

Florea

GL2(Q) automorphic L-functions

Good; Duke, Iwaniec & Sarnak; Kowalski, Michel and vanderKam

Kowalski

Michel

Dirichlet L-functions

Second moment (Selberg, Heath-Brown)

Fourth moment (Heath-Brown, Young)

6th moment of Dirichlet L-functions
averaged over chi and q (C, Iwaniec
and Soundararajan (with a mild t average)
Matomaki and Radziwill (without t average)

8th moment of Dirichlet L-functions
averaged over chi and q
(Chandee, Li, Radziwill)

Soundararajan, upper bounds on RH

Adam Harper, sharp upper bounds on RH

Hughes and Young; mollified 4th moment

Many many other averages

Ratios

Farmer (1995) conjectured that for small $\alpha, \beta, \gamma, \delta \text{ with } \Re\gamma, \Re\delta > 0$

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt$$

$$\sim T\frac{(\alpha+\gamma)(\beta+\delta)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(-\beta+\gamma)(-\alpha+\delta)}{(\alpha+\beta)(\gamma+\delta)}$$

Martin Zirnbauer

Zirnbauer knew that

$$\int_{U(N)} \frac{\Lambda_A(e^{-\alpha})\Lambda_{A^*}(e^{-\beta})}{\Lambda_A(e^{-\gamma})\Lambda_{A^*}(e^{-\delta})} dA$$

$$= \frac{z(\alpha+\beta)z(\gamma+\delta)}{z(\alpha+\delta)z(\beta+\gamma)} + e^{-N(\alpha+\beta)} \frac{z(-\alpha-\beta)z(\gamma+\delta)}{z(-\beta+\delta)z(-\alpha+\gamma)}$$

where

$$\Lambda_A(s) = \prod_{n=1}^{N} (1 - se^{-i\theta_n})$$

and

$$z(x) = \frac{1}{1-e^{-x}}$$

Ratios conjecture (Conrey, Farmer, Zirnbauer; 2007)

Let $\Re\gamma, \Re\delta > 0$ and $\Im\alpha, \Im\beta, \Im\gamma, \Im\delta \ll T^{1-\epsilon}$. Let $s = 1/2 + it$. Then

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt$$

$$= \int_0^T \Bigg(\frac{\zeta(1+\alpha+\beta)\zeta(1+\gamma+\delta)}{\zeta(1+\alpha+\delta)\zeta(1+\beta+\gamma)} A_\zeta(\alpha,\beta,\gamma,\delta) + \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \frac{\zeta(1-\alpha-\beta)\zeta(1+\gamma+\delta)}{\zeta(1-\beta+\delta)\zeta(1-\alpha+\gamma)} A_\zeta(-\beta,-\alpha,\gamma,\delta) \Bigg)\, dt$$

$$+O(T^{1/2+\epsilon})$$

Euler Product

The Euler product A is given by

$$A_\zeta(\alpha,\beta,\gamma,\delta) = \prod_p \frac{\left(1 - \frac{1}{p^{1+\gamma+\delta}}\right)\left(1 - \frac{1}{p^{1+\beta+\gamma}} - \frac{1}{p^{1+\alpha+\delta}} + \frac{1}{p^{1+\gamma+\delta}}\right)}{\left(1 - \frac{1}{p^{1+\beta+\gamma}}\right)\left(1 - \frac{1}{p^{1+\alpha+\delta}}\right)}$$

Leonhard Euler

RATIOS THEOREM (UNITARY)

$$\mathcal{R}(A,B;C,D) := \int_{U(N)} \frac{\prod_{\alpha\in A}\Lambda_X(e^{-\alpha})\prod_{\beta\in B}\Lambda_{X^*}(e^{-\beta})}{\prod_{\gamma\in C}\Lambda_X(e^{-\gamma})\prod_{\delta\in D}\Lambda_{X^*}(e^{-\delta})}dX,$$

with $\Re\gamma > 0, \Re\delta > 0.$

$$Z(A,B) := \prod_{\substack{\alpha\in A\\ \beta\in B}} z(\alpha+\beta),$$

where $z(x) = (1-e^{-x})^{-1}$.

$$Z(A,B;C,D) := \frac{Z(A,B)Z(C,D)}{Z(A,D)Z(B,C)}.$$

Ratios Theorem:

$$\mathcal{R}(A,B;C,D) = \sum_{\substack{S\subset A, T\subset B\\ |S|=|T|}} e^{-N(\sum_{\alpha\in S}\alpha+\sum_{\beta\in T}\beta)} Z(A-S+T^-, B-T+S^-;C,D).$$

RATIOS THEOREM (SYMPLECTIC)

Theorem (CFZ). Suppose $N \geq Q$. Then

$$\int_{USp(2N)} \frac{\prod_{k=1}^{K} \Lambda_A(e^{-\alpha_k})}{\prod_{q=1}^{Q} \Lambda_A(e^{-\gamma_q})} dA$$

$$= \sum_{\varepsilon \in \{-1,1\}^K} e^{N \sum_{k=1}^{K} (\varepsilon_k \alpha_k - \alpha_k)} \frac{\prod_{1 \leq j \leq k \leq K} z(\varepsilon_j \alpha_j + \varepsilon_k \alpha_k) \prod_{1 \leq q < r \leq Q} z(\gamma_q + \gamma_r)}{\prod_{k=1}^{K} \prod_{q=1}^{Q} z(\varepsilon_k \alpha_k + \gamma_q)}.$$

RATIOS THEOREM (ORTHOGONAL)

Theorem (CFZ). Suppose $N \geq Q$. Then

$$\int_{SO(2N)} \frac{\prod_{k=1}^{K} \Lambda_A(e^{-\alpha_k})}{\prod_{q=1}^{Q} \Lambda_A(e^{-\gamma_q})} dA$$

$$= \sum_{\varepsilon \in \{-1,1\}^K} e^{N \sum_{k=1}^{K} (\varepsilon_k \alpha_k - \alpha_k)} \frac{\prod_{1 \leq j < k \leq K} z(\varepsilon_j \alpha_j + \varepsilon_k \alpha_k) \prod_{1 \leq q \leq r \leq Q} z(\gamma_q + \gamma_r)}{\prod_{k=1}^{K} \prod_{q=1}^{Q} z(\varepsilon_k \alpha_k + \gamma_q)}.$$

and

$$\int_{SO(2N+1)} \frac{\prod_{k=1}^{K} \Lambda_A(e^{-\alpha_k})}{\prod_{q=1}^{Q} \Lambda_A(e^{-\gamma_q})} dA$$

$$= \sum_{\varepsilon \in \{-1,1\}^K} \left(\prod_{j=1}^{K} \varepsilon_j\right) e^{(N+1/2) \sum_{k=1}^{K} (\varepsilon_k \alpha_k - \alpha_k)} \frac{\prod_{1 \leq j < k \leq K} z(\varepsilon_j \alpha_j + \varepsilon_k \alpha_k) \prod_{1 \leq q \leq r \leq Q} z(\gamma_q + \gamma_r)}{\prod_{k=1}^{K} \prod_{q=1}^{Q} z(\varepsilon_k \alpha_k + \gamma_q)}.$$

Ratios conjecture (zeta)

Let $Z_\zeta(A,B)=\prod_{\substack{\alpha\in A\\ \beta\in B}}\zeta(1+\alpha+\beta)$ and

$$Z_\zeta(A,B;C,D):=\frac{Z_\zeta(A,B)Z_\zeta(C,D)}{Z_\zeta(A,D)Z_\zeta(B,C)}.$$

Further, let

$$\mathcal{A}_\zeta(A,B;C,D)=\prod_p Z_p(A,B;C,D)\int_0^1 \mathcal{A}_{p,\theta}(A,B;C,D)\,d\theta$$

where $z_p(x):=(1-p^{-x})^{-1}$, $Z_p(A,B)=\prod_{\substack{\alpha\in A\\ \beta\in B}} z_p(1+\alpha+\beta)^{-1}$ and

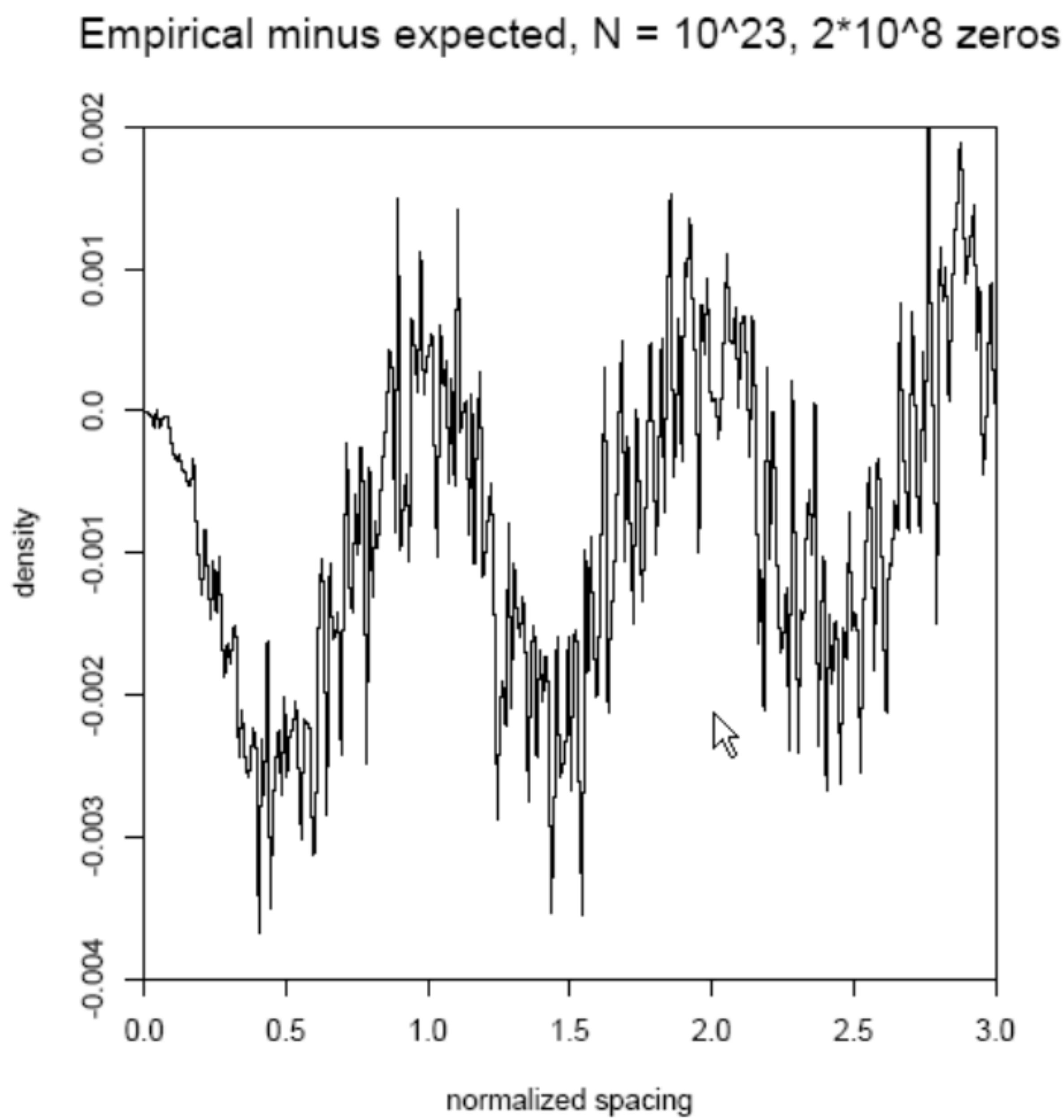
$$Z_p(A,B;C,D):=\frac{Z_p(A,B)Z_p(C,D)}{Z_p(A,D)Z_p(B,C)}$$

and

$$\mathcal{A}_{p,\theta}(A,B;C,D):=\frac{\prod_{\alpha\in A} z_{p,-\theta}(\frac{1}{2}+\alpha)\prod_{\beta\in B} z_{p,\theta}(\frac{1}{2}+\beta)}{\prod_{\gamma\in C} z_{p,-\theta}(\frac{1}{2}+\gamma)\prod_{\delta\in D} z_{p,\theta}(\frac{1}{2}+\delta)}$$

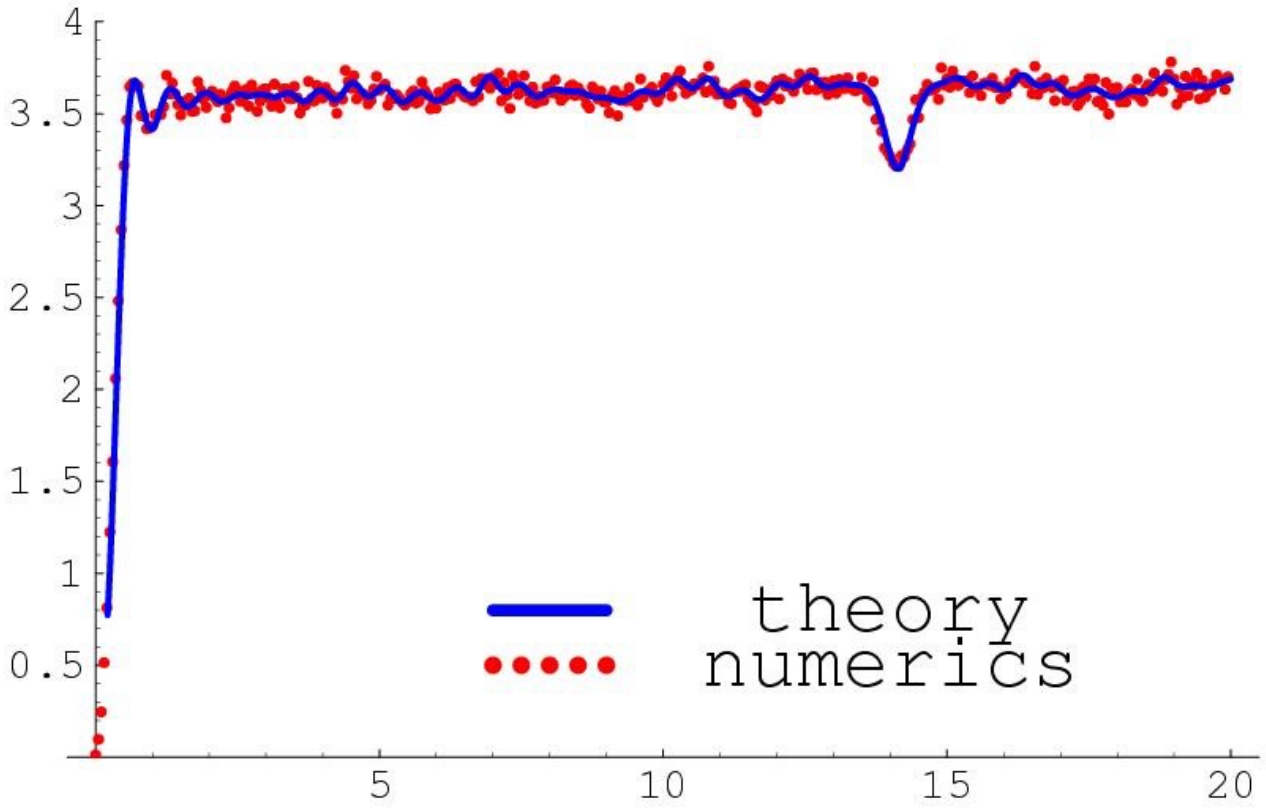
with $z_{p,\theta}(x):=(1-e(\theta)p^{-x})^{-1}$.

Application to pair correlation



Empirical minus expected, N = 10^23, 2*10^8 zeros
density
normalized spacing
0.002
0.001
0.0
-0.001
-0.002
-0.003
-0.004
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Bogomolny and Keating



Eugene Bogomolny

Theorem (Conrey and Snaith 2007) : Assuming a uniform version of the ratios conjecture,

$$\sum_{\gamma,\gamma'}{}' f(\gamma-\gamma') = \frac{1}{(2\pi)^2}\int_0^T f(r)\int_{-T}^{T}\left(\log^2\frac{t}{2\pi} + 2\left(\frac{\zeta'}{\zeta}\right)'(1+ir) + 2\left(\frac{t}{2\pi}\right)^{-ir}\zeta(1-ir)\zeta(1+ir)A(ir) - 2B(ir)\right)\,dr\,dt + O(T^{1/2+\epsilon})$$

where

$$A(\eta) = \prod_p \frac{(1-\frac{1}{p^{1+\eta}})(1-\frac{2}{p}+\frac{1}{p^{1+\eta}})}{(1-\frac{1}{p})^2}$$

$$B(\eta) = \sum_p \left(\frac{\log p}{p^{1+\eta}-1}\right)^2$$

Difference between theory and numerics:

One can do the same approach for pair correlation for RMT, using the ratios theorem. One winds up with

$$\int_{U(N)} \sum_{1\leq j,k\leq N} f(\theta_j,\theta_k)dU_N = \int_0^{2\pi}\int_0^{2\pi} \left(N^2 + J(i\theta_1;-i\theta_2) + J(-i\theta_1;i\theta_2)\right)\ d\theta_1\ d\theta_2$$

where

$$J(a;b) = \left(\frac{z'}{z}\right)'(a+b) + e^{-N(a+b)}z(a+b)z(-a-b),$$

This is

$$\det\begin{pmatrix} N & S_N(u-v) \\ S_N(v-u) & N \end{pmatrix}$$

where recall that

$$S_N(\theta) = \frac{\sin\frac{N\theta}{2}}{\sin\frac{\theta}{2}}$$

Hejhal, 1994 - triple correlation

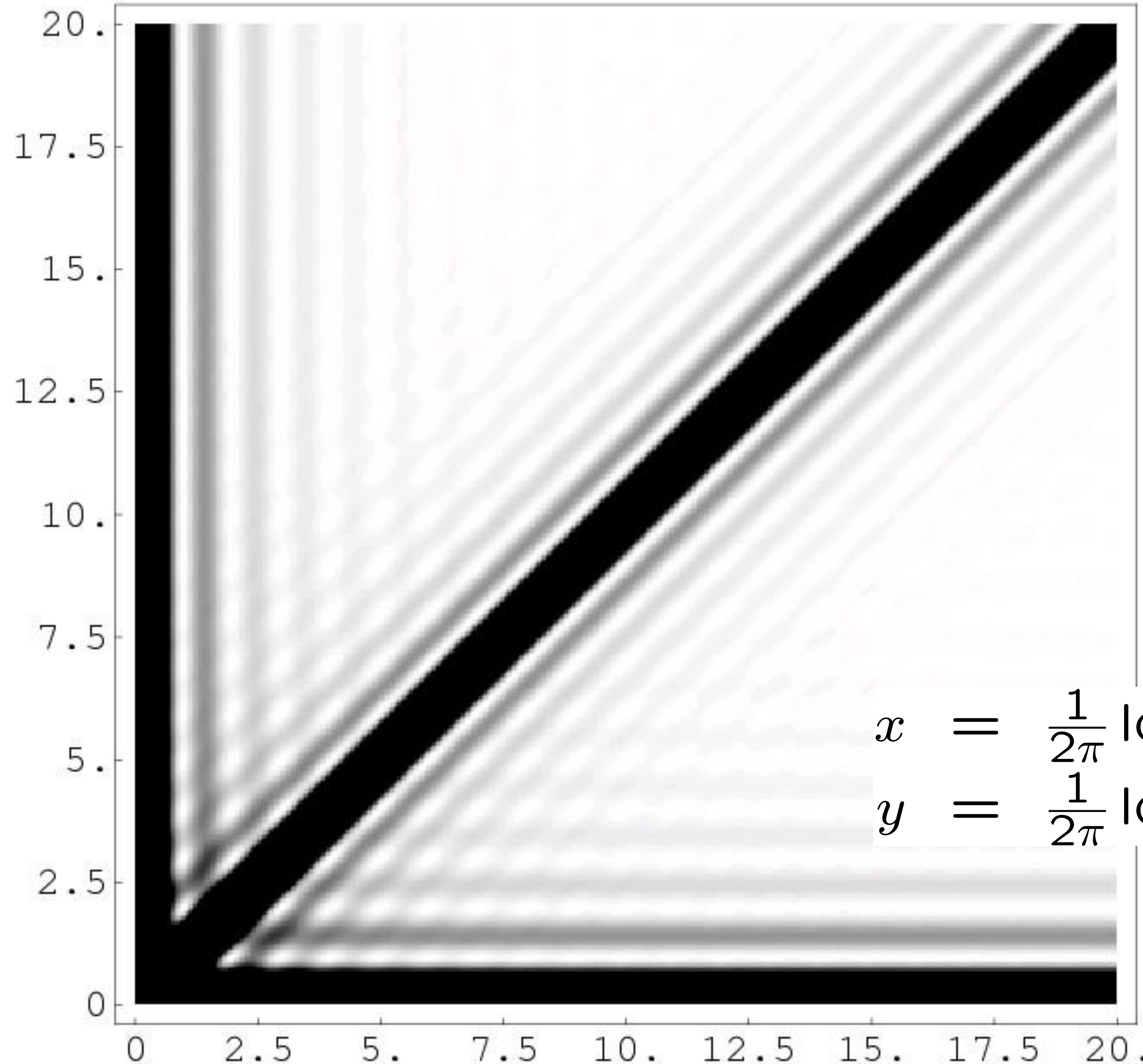
$$\sum_{\substack{\gamma_1,\gamma_2,\gamma_3\in[T,2T] \\ \text{distinct}}} w(\gamma_1,\gamma_2,\gamma_3) f\Big(\frac{\log T}{2\pi}(\gamma_1-\gamma_2), \frac{\log T}{2\pi}(\gamma_1-\gamma_3)\Big)$$

$$= \frac{T\log T}{2\pi}\Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v) \begin{vmatrix} 1 & S(u) & S(v) \\ S(u) & 1 & S(u-v) \\ S(v) & S(u-v) & 1 \end{vmatrix} du\, dv$$

$$+o(1)\Big)$$

where the Fourier transform of f has support on the hexagon with vertices (1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1), and

$$S(u) = \frac{\sin(\pi u)}{\pi u}$$



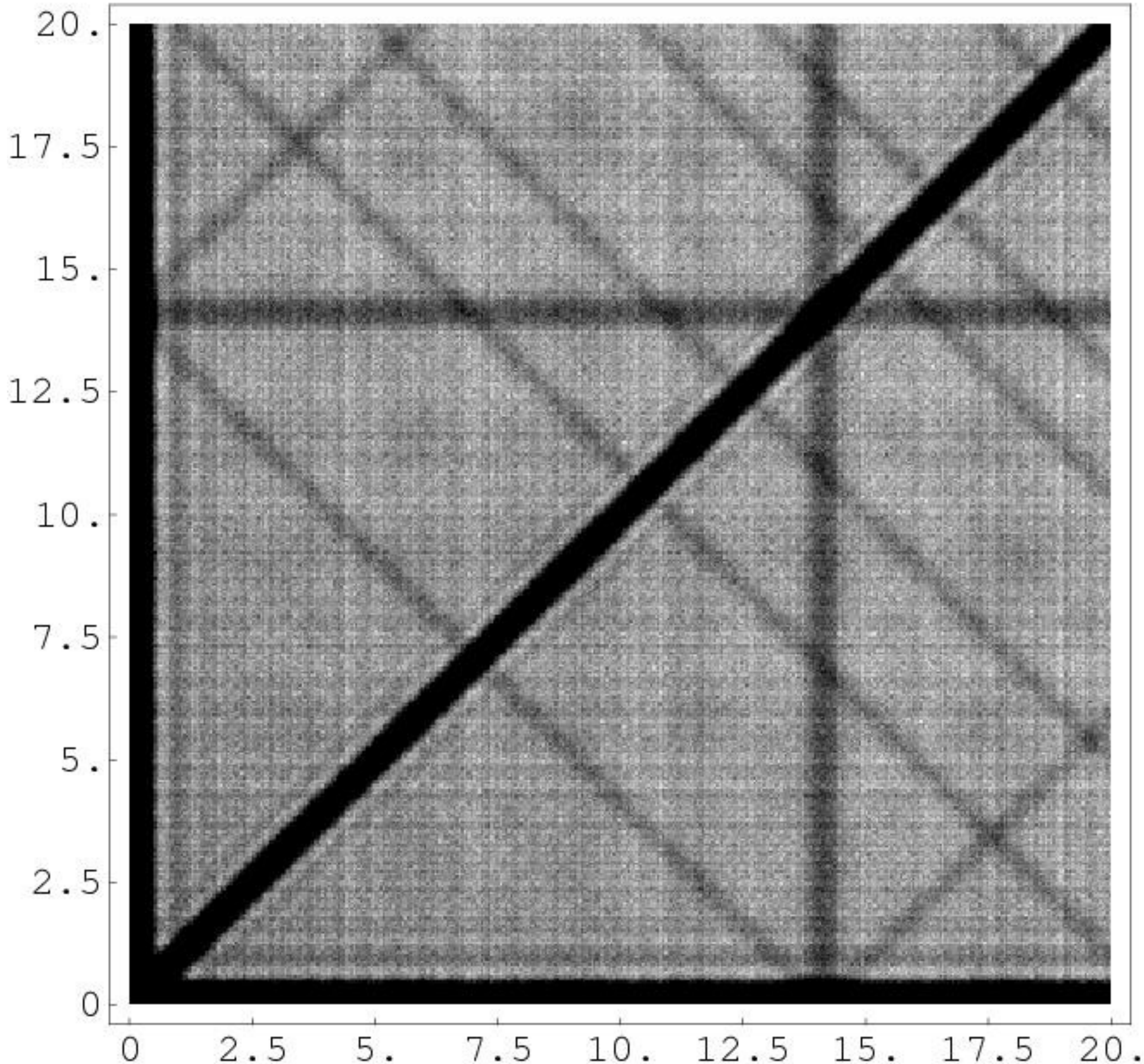
$$x = \frac{1}{2\pi} \log \frac{T}{2\pi}(\gamma_1 - \gamma_2)$$
$$y = \frac{1}{2\pi} \log \frac{T}{2\pi}(\gamma_1 - \gamma_3)$$

RMT triple correlation (N. Snaith)

$$
\begin{aligned}
\sum_{0<\gamma_1,\gamma_2,\gamma_3<T} f(\gamma_1-\gamma_2,\gamma_2-\gamma_3) &= \frac{6\pi^2}{(2\pi)^3}\int_0^T f(0,0)\log\tfrac{t}{2\pi}dt \\
&+\frac{3}{(2\pi)^2}\int_{-T}^{T} f(v,0)\int_0^T \log^2\tfrac{t}{2\pi} + 2\Big(\Big(\frac{\zeta'}{\zeta}\Big)'(1+iv) \\
&\qquad +\Big(\frac{t}{2\pi}\Big)^{-iv}\zeta(1+iv)\zeta(1-iv)A(iv)-B(iv)\Big)dtdv \\
&+\frac{1}{(2\pi)^3}\int_{-T}^{T}\int_{-T}^{T} f(u,v)\int_0^T \log^3\tfrac{t}{2\pi} \\
&\quad +6\log\tfrac{t}{2\pi}\Big(\Big(\frac{\zeta'}{\zeta}\Big)'(1+iu-iv) \\
&\qquad +\Big(\frac{t}{2\pi}\Big)^{-iu+iv}\zeta(1+iu-iv)\zeta(1-iu+iv)A(iu-iv)-B(iu-iv)\Big) \\
&\quad +6\Big(Q(iu,iv)+2\Big(\frac{t}{2\pi}\Big)^{-iu}\zeta(1-iu)\zeta(1+iu) \\
&\qquad \times\Big(A(iu)\Big(\frac{\zeta'}{\zeta}(1+iv-iu)-\frac{\zeta'}{\zeta}(1+iv)\Big)+P(iu,iv)\Big)\Big)dtdudv
\end{aligned}
$$

A,B,Q,P are expressions involving primes

(see Bogomolny, Keating, Phys.Rev.Lett.,1996)



$$x = (\gamma_1 - \gamma_2)$$
$$y = (\gamma_1 - \gamma_3)$$

Zeta triple correlation (N. Snaith)

Applications to lower order terms in one-level densities

Let $L(s,\chi_d)$ be a real, primitive, Dirichlet character of conductor $|d|$. If $1/4 > \Re\gamma > 0$, $|\Re\alpha| < 1/4$, and $\Im\alpha, \Im\gamma \ll X^{1-\epsilon}$, then we conjecture that:

$$\begin{aligned} R_D(\alpha;\gamma) &= \sum_{d\le X} \frac{L(1/2+\alpha,\chi_d)}{L(1/2+\gamma,\chi_d)} = \sum_{d\le X} \Bigg(\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)} A_D(\alpha;\gamma) \\ &\quad + \left(\frac{d}{\pi}\right)^{-\alpha} \frac{\Gamma(1/4-\alpha/2)}{\Gamma(1/4+\alpha/2)} \frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)} A_D(-\alpha;\gamma) \Bigg) + O(X^{1/2+\epsilon}), \end{aligned}$$

where

$$A_D(\alpha;\gamma) = \prod_p \left(1 - \frac{1}{p^{1+\alpha+\gamma}}\right)^{-1} \left(1 - \frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\gamma}}\right).$$

Assuming the ratios conjecture and that $\frac{1}{\log X} \ll \Re r < \frac{1}{4}$ and $\Im r \ll X^{1-\epsilon}$ we have

$$\sum_{d \le X} \frac{L'(1/2+r,\chi_d)}{L(1/2+r,\chi_d)}$$
$$= \sum_{d \le X} \Big(\frac{\zeta'(1+2r)}{\zeta(1+2r)} + A_D'(r;r) - \left(\frac{d}{\pi}\right)^{-r} \frac{\Gamma(1/4-r/2)}{\Gamma(1/4+r/2)} \zeta(1-2r) A_D(-r;r)\Big)$$
$$+ O(X^{1/2+\epsilon}),$$

One-level density (C and Snaith)

Assuming the ratios conjecture, we have

$$\sum_{d\le X}\sum_{\gamma_d} f(\gamma_d) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) \sum_{d\le X} \Big(\log\frac{d}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4+it/2) + \frac{1}{2}\frac{\Gamma'}{\Gamma}(1/4-it/2) +$$
$$2\Big(\frac{\zeta'(1+2it)}{\zeta(1+2it)} + A_D'(it;it) - \left(\frac{d}{\pi}\right)^{-it}\frac{\Gamma(1/4-it/2)}{\Gamma(1/4+it/2)}\zeta(1-2it)A_D(-it;it)\Big)\Big)\,dt$$
$$+O(X^{1/2+\epsilon}),$$

where

$$A_D(-r;r) = \prod_p \left(1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1}\right)\left(1-\frac{1}{p}\right)^{-1},$$

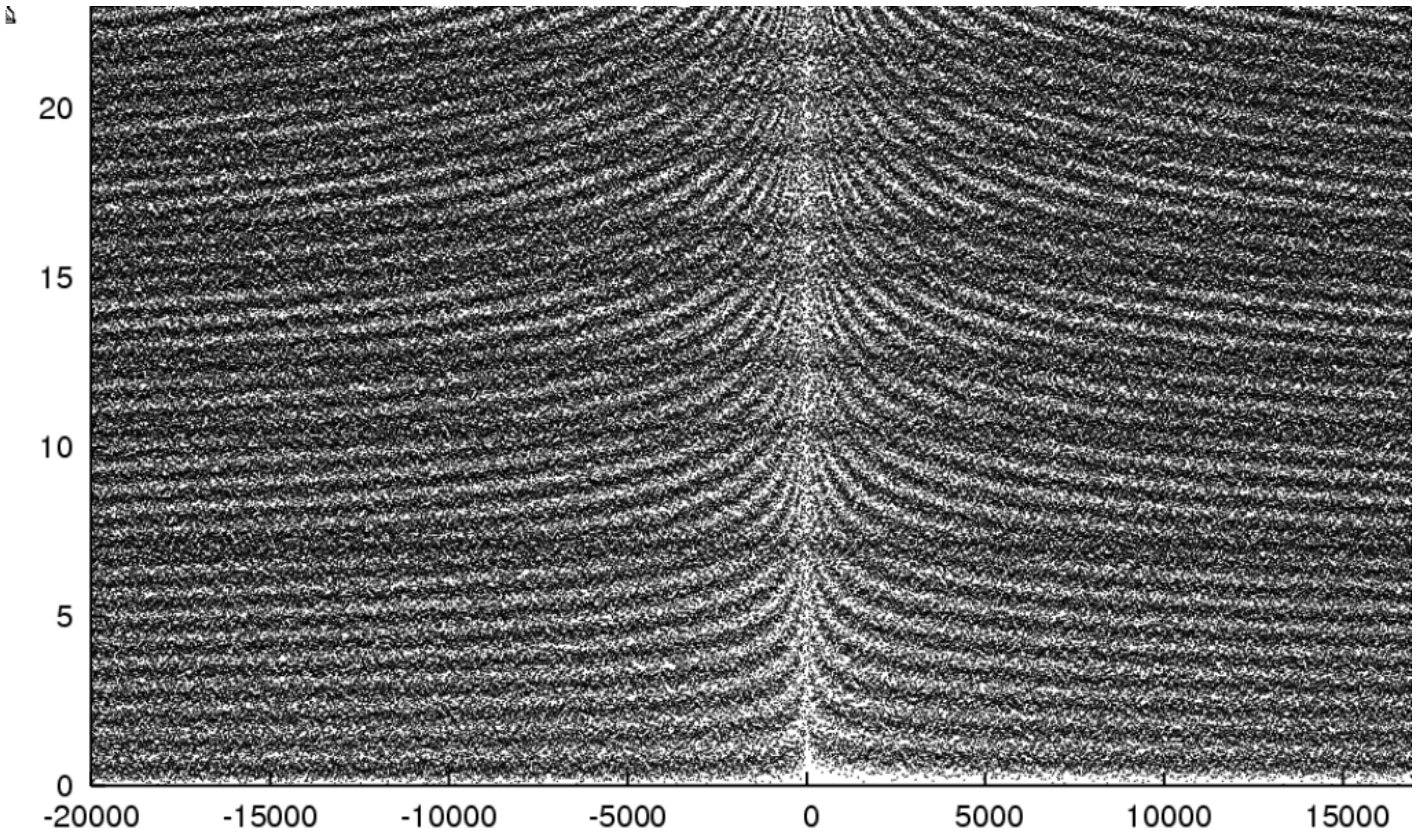
and

$$A_D'(r;r) = \sum_p \frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Michael Rubinstein and Pang Gao
investigated n-level densities for this family.

Mike Rubistein

Zeros of quadratic L-functions



by Mike Rubinstein

The ratios conjecture implies
the theta = infinity conjecture,
which in turn implies RH

Steve Gonek

Bettin, Gonek (2017)
"The $\theta=\infty$ conjecture implies
the Riemann hypothesis"

Sandro Bettin

Other density results

Rubinstein (2001): n-level density for quadratic L-functions, support of f<1

Gao (2005): n-level density for quadratic L-functions, support of f<1

Entin, Roddity Gershon, Rudnick (2013)

combinatorics of n-level for quadratic
L-functions using function fields

Mason and Snaith (2016)

combinatorics of n-level for orthogonal and
symplectic L-families using RMT

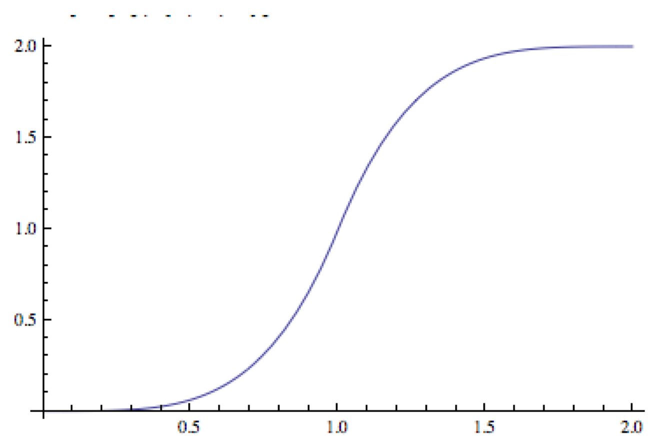
> 20 papers on the arxiv with "low-lying zeros" in the title since 1995
by Alpoge, Amersi, Baier, Chandee, Cho, Duenez, Fiorilli, Gao, Hughes, Iyer, Lazarov, Lee, Levinson, Liu, Mackall, Miller, Park, Parks, Peckner, Rapti, Ricotta, Royer, Shin, Sodergren, Templier, Turner-Butterbaugh, Winsor, Young, Zhang, Zhao

Moments of long Dirichlet polynomials

$$\int_0^T \left| \sum_{n \le X} \frac{d(n)}{n^{1/2+it}} \right|^2 dt \sim M_2(\alpha) \frac{T}{2\pi^2} \log^4 T$$

where $X = T^\alpha$ and

$$M_2(\alpha) = \begin{cases} \alpha^4 & \text{if } 0 < \alpha < 1 \\ -\alpha^4 + 8\alpha^3 - 24\alpha^2 + 32\alpha - 14 & \text{if } 1 < \alpha < 2 \\ 2 & \text{if } \alpha > 2 \end{cases}$$



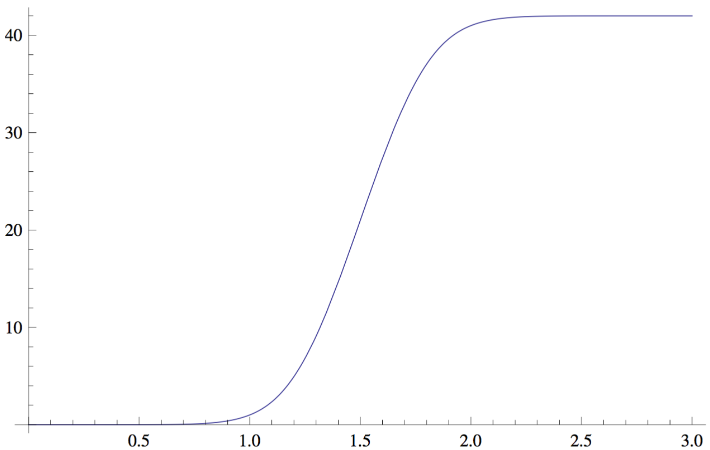


FIGURE 2. The plot of $M_3(\alpha)$ for $0 < \alpha < 3$.

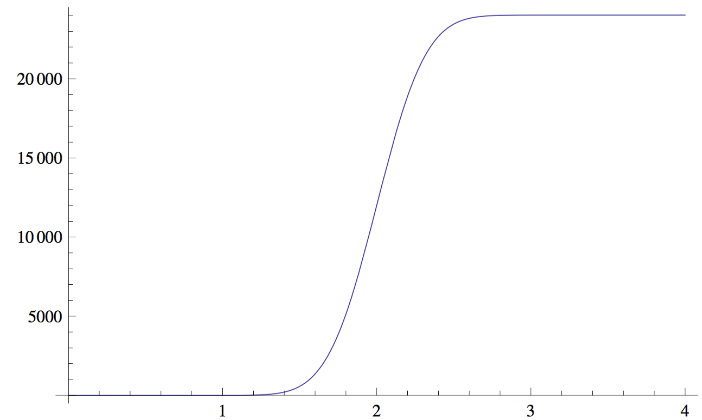


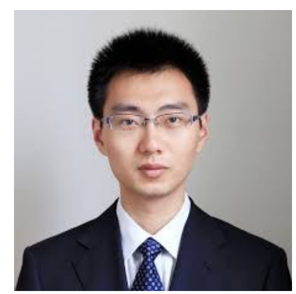
FIGURE 3. The plot of $M_4(\alpha)$ for $0 < \alpha < 4$.

Goldston and Gonek (1998)
“Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series”

Keating, Roddity-Gershon, Rodgers, Rudnick
“Sums of divisor functions in Fq[t] and matrix integrals” (2018)

Rodgers, Soundararajan (2018)
“The variance of divisor sums in arithmetic progressions”

Basor, Ge, and Rubinstein (2018)
“Some multidimensional integrals in number and connections with the Painleve V equation”



Bettin, private communication (2017)

Conrey, Keating (2015-2018)
“Moments of zeta and divisor correlations, I - V”

Let

$$\zeta(s)^k = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

and

$$\Lambda_U(s)^k = \sum_{n \leq Nk} \delta_{k,U}(n) s^n$$

We conjecture that

$$\frac{1}{TL^{k^2}a_k}\int_0^T \left| \sum_{n\le T^\alpha} \frac{d_k(n)}{n^{1/2+it}} \right|^2 \, dt$$

and

$$\frac{1}{N^{k^2}}\int_{U(N)} \left| \sum_{n\le \alpha N} \delta_{k,U}(n) \right|^2 \, dU$$

are asymptotically equal.

Bettin proved that the moment conjecture implies this.

However

$$\frac{1}{N^{k^2}} \int_{U(N)} \left| \sum_{n \leq \alpha N} \delta_{k,U}(n) \right|^2 \, dU$$

$$= \frac{1}{N^{k^2}} \int_{U(N)} \sum_{n \leq \alpha N} \left| \delta_{k,U}(n) \right|^2 \, dU$$

whereas

$$\frac{1}{TL^{k^2}a_k}\int_0^T \left| \sum_{n\le T^\alpha} \frac{d_k(n)}{n^{1/2+it}} \right|^2 \, dt$$

is not asymptotic to

$$\frac{1}{TL^{k^2}a_k}\int_0^T \sum_{n\le T^\alpha} \left| \frac{d_k(n)}{n^{1/2+it}} \right|^2 \, dt$$

Divisor correlations

We need information about

$$\sum_{n \leq X} \tau_A(n) \tau_B(n+h)$$

The delta method of Duke, Friedlander and Iwaniec (1993) can provide the needed conjecture.

Bill Duke

John Friedlander

Henryk Iwaniec

Delta method conjecture

$$\langle \tau_A(m)\tau_B(n)\rangle_{m=u}^{(*)}$$
$$\sim \frac{1}{M}\sum_{q=1}^{\infty} r_q(h)\langle \tau_A(m)e(mN/q)\rangle_{m=u}\langle \tau_B(n)e(nM/q)\rangle_{n=\frac{uN}{M}}$$

where $(*) : mM - nN = h$

$$\langle \tau_A(m)e(mN/q)\rangle_{m=u} = \frac{1}{2\pi i}\int_{|w-1|=\epsilon} D_A(w, e(\frac{N}{q}))u^{w-1}\, dw$$

where

$$D_A(w, e(\frac{N}{q})) = \sum_{n=1}^{\infty} \frac{\tau_A(n)e(nN/q)}{n^w}.$$

The poles of this Dirichlet series can be determined by replacing the exponential by Dirichlet characters and finding the coefficient of the trivial character (i.e. zeta).

Assuming delta-conjecture

$$\int_0^\infty \psi(\frac{t}{T}) \sum_{\substack{m\le T^r \\ n \le T^r}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} \left(\frac{m}{n}\right)^{it} \; dt$$

$$= \int_0^\infty \psi(\frac{t}{T}) \Bigg(\mathcal{B}_{A,B}(1;T^r)$$

$$+ \sum_{\substack{\alpha\in A \\ \beta\in B}} \left(\frac{t}{2\pi}\right)^{-\alpha-\beta} \mathcal{B}_{A',B'}(1;T^r) \Bigg)$$

$+O(T^{1-\delta})$ where $1 \le r < 2$ and

$$A' = A - \{\alpha\} \cup \{-\beta\}, B' = B - \{\beta\} \cup \{-\alpha\}$$

What if $r > 2$?

Say $\ell < r < \ell + 1$

Identity

Suppose that

$$A = A_1 \cup \cdots \cup A_\ell$$

and

$$B = B_1 \cup \cdots \cup B_\ell$$

Then

$$\sum_{m=n} \frac{\tau_A(m)\tau_B(n)}{m^s n^z}$$

$$= \sum_{\substack{M_1 \ldots M_\ell = N_1 \ldots N_\ell \\ (M_i, N_i)=1}} \prod_{j=1}^{\ell} \left(\sum_{M_j m_j = N_j n_j} \frac{\tau_{A_j}(m_j)\tau_{B_j}(n_j)}{m_j^s n_j^z} \right)$$

Conrey - Keating approach

$$\sum_{m,n<T^r} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} \hat{\psi}(\frac{T}{2\pi}\log\frac{m}{n})$$

is related to

$$\sum_{\substack{A=A_!\cup\cdots\cup A_\ell \\ B=B_1\cup\cdots\cup B_\ell}} \sum_{\substack{M_1\ldots M_\ell=N_1\ldots N_\ell \\ (M_i,N_i)=1}} \prod_{j=1}^{\ell} \left(\sum_{m_j,n_j} \frac{\tau_{A_j}(m_j)\tau_{B_j}(n_j)}{\sqrt{mn}} \right) \hat{\psi}(\frac{T}{2\pi}\log\frac{m_1\ldots m_\ell}{n_1\ldots n_\ell})$$

We need to average over

$$* \left\{ \begin{array}{ccc} M_1 m_1 & = & N_1 n_1 + h_1 \\ & \ldots & \\ M_\ell m_\ell & = & N_\ell n_\ell + h_\ell \end{array} \right\}$$

weighted by divisor coefficients. Note that

$$\hat{\psi}(\frac{T}{2\pi}\log\frac{m_1\ldots m_\ell}{n_1\ldots n_\ell}) \sim \hat{\psi}(\frac{T}{2\pi}\sum\frac{h_i}{n_i N_i})$$

Connecting divisor correlations and the recipe

$$\frac{1}{2\pi i}\int_{(2)} X^s\Big(\frac{T}{2\pi}\Big)^{-\ell s}\sum_{\substack{(M_1,N_1)=\cdots=(M_\ell,N_\ell)=1\\ N_1\cdots N_\ell=M_1\cdots M_\ell\\ \epsilon_j\in\{-1,+1\}}}\int_{0<v_1,\ldots,v_\ell<\infty}\hat{\psi}(\epsilon_1 v_1+\cdots+\epsilon_\ell v_\ell)$$

$$\prod_{j=1}^{\ell}\Bigg[\frac{1}{(2\pi i)^2}\iint_{\substack{|w_j-1|=\epsilon\\ |z_j-1|=\epsilon}} M_j^{-z_j}N_j^{s+1-w_j}\sum_{h_j,q_j}\frac{r_{q_j}(h_j)}{h_j^{s+2-w_j-z_j}}v_j^{s+1-w_j-z_j}$$

$$D_{A_j}(w_j,e(\frac{N_j}{q_j}))D_{B_j}(z_j,e(\frac{M_j}{q_j}))(\frac{T}{2\pi})^{w_j+z_j-2}dw_jdz_j\ dv_j\Bigg]\ \frac{ds}{s}$$

$$= \quad \frac{1}{2\pi i}\int_{(2)} X^s\int_0^\infty \psi(t)\sum_{\substack{U(\ell)\subset A\\ V(\ell)\subset B}}\left(\frac{Tt}{2\pi}\right)^{-\sum_{\substack{\hat\alpha\in U(\ell)\\ \hat\beta\in V(\ell)}}(\hat\alpha+\hat\beta+s)}$$

$$\times\mathcal{B}(A_s-U(\ell)_s+V(\ell)^-,B-V(\ell)+U(\ell)_s{}^-,1)\ dt\ \frac{ds}{s}$$

where $U(\ell)$ denotes a set of cardinality ℓ with precisely one element from each of $A_1,\ldots,A_\ell$ and similarly $V(\ell)$ denotes a set of cardinality ℓ with precisely one element from each of $B_1,\ldots,B_\ell$.

Automorphisms

If we sum this over all the ways to split up A and B we get what the recipe predicts times a factor

$$\ell^{2k-2\ell}$$

But this is the number of automorphisms of the *-system.

Wooley has pointed out the connection with counting points on varieties and Manin's idea of counting points on certain varieties by counting points on a stratified set of subvarieties; this idea may be relevant here.

Trevor Wooley

Y. I.Manin

Symplectic Identity

If $\quad A = A_1 \cup \cdots \cup A_\ell$

then

$$\sum_{r=\square} \frac{\tau_A(r)}{r^s} = \sum_{M_1 \ldots M_\ell = \square} \prod_{j=1}^{\ell} \mu^2(M_j) \sum_{M_j r_j = \square} \frac{\tau_{A_j}(r_j)}{r_j^s}$$

Ranks of elliptic curves

Which integers m are the sum of two rational cubes?

Which integers m are the sum of two rational cubes?

$$6 = \left(\frac{37}{21}\right)^3 + \left(\frac{17}{21}\right)^3$$

346 is the sum of two rational cubes

$$346 = \left(\frac{4718903581349993258016910385678696432159277067}{810669511745132570258197805629318670369406473 5}\right)^3 + \left(\frac{4297900568569819370828623372794159538252654468 3}{8106695117451325702581978056293186703694064735}\right)^3$$

Table 6.

The number g of generators and the basic solutions of the equation $X^3 + Y^3 = AZ^3$, A cubefree and ≤ 500.

A	g	(X, Y, Z)
6	1	(37, 17, 21)
7	1	(2, −1, 1)
9	1	(2, 1, 1)
12	1	(89, 19, 39)
13	1	(7, 2, 3)
15	1	(683, 397, 294)
17	1	(18, −1, 7)
19	2	(3, −2, 1), (5, 3, 2)
20	1	(19, 1, 7)
22	1	(25 469, 17 299, 9 954)
26	1	(3, −1, 1)
28	1	(3, 1, 1)
30	2	(163, 107, 57), (289, −19, 93)
31	1	(137, −65, 42)
33	1	(1 853, 523, 582)
34	1	(631, −359, 182)
35	1	(3, 2, 1)
37	2	(4, −3, 1), (10, −1, 3)
42	1	(449, −71, 129)
43	1	(7, 1, 2)
49	1	(11, −2, 3)
50	1	(23 417, −11 267, 6 111)
51	1	(72[illegible] 511, 62 641, 197 028)
53	1	(1 872, −1 819, 217)
58	1	(28 747, −14 653, 7 083)
61	1	(5, −4, 1)
62	1	(11, 7, 3)
63	1	(4, −1, 1)
65	2	(4, 1, 1), (191, −146, 39)
67	1	(5 353, 1 208, 1 323)
68	1	(2 538 163, −472 663, 620 505)
69	1	(15 409, −10 441, 3 318)
70	1	(53, 17, 13)
71	1	(197, −126, 43)
75	1	(17 351, −11 951, 3 606)
78	1	(5 563, 53, 1 302)
79	1	(13, −4, 3)
84	1	(433, 323, 111)
85	1	(2 570 129, −2 404 889, 330 498)
86	2	(13, 5, 3), (10 067, −10 049, 399)
87	1	(1 176 498 611, −907 929 611, 216 266 610)
89	1	(53, 36, 13)
90	1	(1 241, −431, 273)
91	2	(4, 3, 1), (6, −5, 1)
92	1	(25 903, −3 547, 5 733)
94	1	(15 642 626 656 646 177, −15 616 184 186 396 177, 590 736 058 375 050)
97	1	(14, −5, 3)
98	1	(5, −3, 1)
103	1	(592, −349, 117)
105	1	(4 033, 3 527, 1 014)
106	1	(165 889, −140 131, 25 767)
107	1	(90, 17, 19)
110	2	(181, −71, 37), (629, 251, 134)
114	1	(9 109, −901, 1 878)
115	1	(5 266 097, −2 741 617, 1 029 364)
117	1	(5, −2, 1)
123	1	(184 223 499 139, 10 183 412 861, 37 045 412 880)
124	2	(5, −1, 1), (479, −443, 57)
126	2	(5, 1, 1), (71, −23, 14)
127	2	(7, −6, 1), (121, −120, 7)
130	1	(52 954 777, 33 728 183, 11 285 694)
132	2	(2 089, −901, 399), (39 007, −29 503, 6 342)
133	1	(5, 2, 1)
134	1	(9, 7, 2)
139	1	(16, −7, 3)
140	1	(27 397, 6 623, 5 301)
141	1	(53 579 249, −52 310 249, 4 230 030)
142	1	(2 454 839, 1 858 411, 530 595)
143	1	(73, 15, 14)
151	1	(338, −95, 63)
153	2	(70, −19, 13), (107, −56, 19)
156	1	(2 627, −1 223, 471)
157	1	(19 964 887, −19 767 319, 1 142 148)
159	1	(103 750 849, 2 269 079, 19 151 118)
161	[illegible]	(39, −16, 7)
163	2	(11, −3, 2), (17, −8, 3)
164	1	(311 155 001, −236 283 589, 46 913 867)
166	1	(1 374 582 733 040 071, −1 295 038 816 428 439, 136 834 628 063 958)

Selmer's table of A for which

$$x^3 + y^3 = A$$

has infinitely many solutions

If m is squarefree and $m \equiv 4, 7, 8$ mod 9 then it is believed that there will always be a solution of $x^3 + y^3 = m$. If m is 1, 2, or 5 mod 9, then solutions are believed to be rare.

Conjecture from random matrix theory:

"Rare" solutions with m congruent to 2 mod 7 are exactly twice as likely as rare solutions with m congruent to 3 mod 7.

Watkins computed data for m up to ten million. What he found suggests that there are 125728 values of m congruent to 2 modulo 7 and only 59440 m congruent to 3 modulo 7 for which $x^3 + y^3 = m$ has a solution (m restricted to 1, 2, or 5 mod 9).

$$\frac{125728}{59440} = 2.11$$

Mark Watkins

This conjecture depends fundamentally on RMT!

Moments of characteristic polynomials (Keating and Snaith)

$$
\begin{aligned}
M_{U,N}(s) &= \int_{U(N)} |\det(A-I)|^s \, dA \\
&= \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{\Gamma(j+s/2)^2} ,
\end{aligned}
$$

$$
\begin{aligned}
M_{Sp,2N}(s) &= \int_{Sp(2N)} |\det(A-I)|^s \, dA \\
&= 2^{2Ns} \prod_{j=1}^{N} \frac{\Gamma(1+N+j)\Gamma(1/2+s+j)}{\Gamma(1/2+j)\Gamma(1+s+N+j)} ,
\end{aligned}
$$

$$
\begin{aligned}
M_{O,2N}(s) &= \int_{O(2N)} |\det(A-I)|^s \, dA \\
&= 2^{Ns} \prod_{j=1}^{N} \frac{\Gamma(N+j-1)\Gamma(s+j-1/2)}{\Gamma(j-1/2)\Gamma(s+j+N-1)} .
\end{aligned}
$$

CKRS, Watkins

By random matrix theory (using discretization and the complex moments of characteristic polynomials of orthogonal matrices) we expect rank two curves to occur in the family of quadratic twists of E for about

$$C_E x^{3/4} (\log x)^{b_E}$$

values of $d < x$ where C_E and b_E are certain constants:

$$b_E = \begin{cases} 11/8 & \text{if 3 order 2 torsion pts} \\ \sqrt{2} - 5/8 & \text{if 1 order 2 torsion pt.} \\ 3/8 & \text{if 0 order 2 torsion pts.} \end{cases}$$

If we restrict to prime discriminants, then the power on the log should be $-5/8$ for any curve.

How are elliptic curves of rank 2 from the family of twists of a fixed E, distributed in arithmetic progressions?

Fix a prime q; consider

$$R_q(X) = \frac{\sum_{\substack{|d|<X, w_E\chi_d(-Q)=1 \\ L_E(1,\chi_d)=0 \\ \chi_d(q)=1}} 1}{\sum_{\substack{|d|<X, w_E\chi_d(-Q)=1 \\ L_E(1,\chi_d)=0 \\ \chi_d(q)=-1}} 1}$$

Conjecture: (C, Keating, Rubinstein, Snaith; 2000)

$$\lim_{X\to\infty} R_q(X) = R_q := \left(\frac{q+1-a_q}{q+1+a_q}\right)^{1/2}.$$

p	conjectured R_p for E_{11}	data for E_{11}	conjectured R_p for E_{19}	data for E_{19}	conjectured R_p for E_{32}	data for E_{32}
3	1.2909944	1.2774873	1.7320508	1.7018241	1	0.99925886
5	0.84515425	0.84938811	0.57735027	0.57825622	1.4142136	1.4113424
7	1.2909944	1.288618	1.1338934	1.134852	1	1.0003445
11		0	0.77459667	0.76491219	1	1.0001457
13	0.74535599	0.73266305	1.3416408	1.3632977	0.63245553	0.61626177
17	1.118034	1.1282072	1.183216	1.196637	0.89442719	0.88962298
19	1	1.000864		0	1	1.0006726
23	1.0425721	1.0470095	1	0.99857962	1	1.0000812
29	1	0.99769402	0.81649658	0.80174375	1.4142136	1.4615854
31	0.80064077	0.78332934	1.1338934	1.143379	1	1.0008405
37	0.92393644	0.91867671	0.9486833	0.94311279	1.0540926	1.0603105
41	1.2126781	1.2400086	1.1547005	1.1683113	0.78446454	0.76494748
43	1.1470787	1.1642671	1.0229915	1.0229106	1	1.0006774
47	0.84515425	0.82819492	1.0645813	1.0708874	1	0.99951502
53	1.118034	1.1332312	0.79772404	0.77715638	0.76696499	0.74137107
59	0.91986621	0.91329134	1.1055416	1.1196252	1	0.99969828
61	0.82199494	0.79865031	1.0162612	1.0199932	1.1766968	1.1996892
67	1.1088319	1.1216776	1.0606602	1.0705574	1	1.0002831
71	1.0425721	1.0497774	0.91986621	0.90939741	1	0.99992715
73	0.94733093	0.94345043	1.099525	1.1110782	1.0846523	1.0950853
79	1.1338934	1.1562237	0.90453403	0.8922209	1	0.99882039
83	1.0741723	1.0854551	0.8660254	0.84732408	1	0.99979996
89	0.84515425	0.82410673	0.87447463	0.85750248	0.89442719	0.88154899
97	1.0741723	1.0877289	0.92144268	0.90867892	0.8304548	0.80811684
101	0.98058068	0.97846254	0.94280904	0.93032086	1.0198039	1.0229108
103	1.1677484	1.1976448	0.87333376	0.855721	1	1.0004009
107	0.84515425	0.82186438	1.183216	1.2153554	1	1.0009282
109	0.91287093	0.89933354	1.1577675	1.1844329	0.94686415	0.94015124
113	0.92393644	0.9146531	0.9486833	0.93966595	1.1313708	1.1534106
127	0.93933644	0.93052596	0.98449518	0.98005032	1	0.99904006
131	1.1470787	1.171545	1.1208971	1.1413931	1	0.99916309
137	1.052079	1.0603352	1.0219806	1.0285831	1.1744404	1.2066518

Higher “ranks” in the family of quadratic twists of a weight 4 or 6 modular forms

For a weight 4 form f we expect that $C_f x^{1/4} (\log x)^{b_f}$ quadratic twists $d < x$ will vanish at the central point.

For a weight ≥ 6 form f we expect that only a bounded number of quadratic twists $d < x$ will vanish at the central point.

3	1.18	1.11
5,	0.55	0.59
11,	1.06	1.15
13,	0.86	0.84
17,	0.84	0.76
19,	1.35	1.53
23,	0.92	0.87
29,	1.14	1.22
31,	0.99	1.05
37,	1.19	1.17
41,	0.90	0.82
43,	0.93	0.90
47,	0.87	0.76
53,	1.06	1.06
59,	0.79	0.75
61,	1.14	1.15
67,	0.95	0.94
71,	1.16	1.17
73,	1.14	1.08
79,	0.93	0.93
83,	1.21	1.18
89,	0.91	0.87
97,	0.97	0.98

Vanishings of twists of the level 7 weight 4 cusp form.

There are 1155 vanishings out of 13298378 twists up to d=100,000,000

The first column is the prime, the second is the random matrix prediction; the last is the data.

The RMT prediction is

$$\sqrt{\frac{p^2+p+a_p}{p^2+p-a_p}}$$

Twists of a weight 2 form by a cubic Dirichlet character

Work of David, Fearnley, and Kivilevsky (2004).

They obtain $\gg x^{1/2-\epsilon}$ vanishing twists for conductors $< x$. RMT predicts the number will be $\sim C_E x^{1/2}(\log x)^{b_E}$

For twists by characters of order 5 they expect the number of vanishings goes to ∞ slowly.

For twists by characters of order 7 and greater they expect a bounded number will vanish.

The RMT model involves a unitary model.

(Watkins)

Suggestion for the frequency of rank 3 vanishing

$$\#\{d \leq X : L_E(s,\chi_d) \text{ has a triple zero } \} \gg x^{3/4-\epsilon}$$

Might be plausible based on Elkies data for rank 3 curves among twists of the congruent number curve. RMT suggests

$$\lim_{X\to\infty} \frac{\#\{d \leq X : L_E(s,\chi_d) \text{ has a triple zero}, d \equiv \text{ square } \bmod p\}}{\#\{d \leq X : L_E(s,\chi_d) \text{ has a triple zero}, d \equiv \text{non-square } \bmod p\}}$$
$$= \left(\frac{p+1-a_p}{p+1+a_p}\right)^{-3/2}$$

$$\eta(4z)^2\eta(8z)^2 = q - 2q^5 - 3q^9 + 6q^{13} + 2q^{17} + -q^5 - 10q^{29} - 2q^{37} + 10q^{41} + \dots$$

Elkies data about rank 3 twists of the congruent number curve, sorted by Watkins

The first column is the prime.

The second column is the number of rank 3's in square residue classes.

The third column is the number of rank 3's in non-square classes.

The fourth column is the ratio of columns two and three.

The last column is the RMT prediction.

3	2705	2867	0.943495	1
5	4240	1951	2.17324	2.82843
7	3281	3276	1.00153	1
11	3698	3731	0.991155	1
13	1827	5580	0.327419	0.252982
17	3186	4197	0.759114	0.715542
19	3943	3998	0.986243	1
23	3899	3947	0.987839	1
29	5873	2249	2.61138	2.82843
31	4032	4083	0.987509	1
37	4451	3820	1.16518	1.17121
41	2711	5411	0.501016	0.482747
43	4202	4165	1.00888	1
47	4219	4091	1.03129	1
53	2672	5723	0.466888	0.451156
59	4266	4240	1.00613	1
61	5239	3245	1.61448	1.62927
67	4226	4306	0.981421	1
71	4166	4289	0.971322	1
73	4696	3688	1.27332	1.27606
79	4221	4246	0.994112	1
83	4326	4253	1.01716	1
89	3648	4828	0.755592	0.715542

Let $E_{a,b}$ denote the elliptic curve $y^2 = x^3 + ax + b$.

Define a family by

$$\mathcal{F}'(X) := \{E_{a,b^2} : |a| \le X^{1/3}, |b| \le X^{1/4}, p^4 \mid a \rightarrow p^3 \nmid b\}$$

Matt Young's conjecture (2010): If $|\Re\alpha| < 1/6$ then

$$\sum_{E\in\mathcal{F}'(X)} L(1/2+\alpha, E) = \frac{A(\alpha)}{\zeta(1+\alpha)}(1 + O(N_E^{-\delta}))$$

where A is an absolutely convergent Euler product.

Matt Young

Corollary: $\zeta(s) \neq 0$ for $\Re s > 5/6$.

Extreme Values

Conjecture: Farmer, Gonek, Hughes (2006)

$$\max_{t\in[0,T]} |\zeta(1/2+it)| = \exp\left((1+o(1))\sqrt{\frac{1}{2}\log T \log\log T}\right)$$

What RMT won't do

Constants that involve primes

Main terms of size $x^{1-\delta}$

Shifted convolution problems

What number theory hasn't done

First moment of zeta

Complete main terms for real moments

RH

>From: Enrico Bombieri <eb@IAS.EDU>
Date: Tue, 1 Apr 1997 12:35:12 -0500
To: eb@IAS.EDU, zeilberg@euclid.math.temple.edu

Dear Doron,

There are fantastic developments to Alain Connes's lecture at IAS last Wednesday. Connes gave an account of how to obtain a trace formula involving zeroes of L-functions only on the critical line, and the hope was that one could obtain also Weil's explicit formula in the same context; this would solve the Riemann hypothesis for all L-functions at one stroke. Thus ther cannot be even a single zeroe(1) off the critical line!

Well, a young physicist at the lecture saw in a flash that one could set the whole thing in a combinatorial setting using supersymmetric fermionic-bosonic systems (the physics corresponds to a near absolute zero ensemble of a mixture of anyons and morons with opposite spins) and, using the C-based meta-language MISPAR, after six days of uninterrupted work, computed the logdet of the resolvent Laplacian, removed the infinities using renormalization, and, lo and behold, he got the required positivity of Weil's explicit formula! Wow!

Regards also from Paula Cohen.
Please give this the highest diffusion. Best,

Enrico

The End