Positivity of L-functions and "Completion of square"

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Outline

1. Riemann hypothesis
2. Positivity of L-functions
3. Completion of square
4. Positivity on surfaces
Riemann Hypothesis (RH)

Riemann zeta function

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \]

\[ = \prod_{p, \text{ primes}} \frac{1}{1 - p^{-s}}, \quad s \in \mathbb{C}, \ \text{Re}(s) > 1 \]

Analytic continuation and Functional equation

\[ \Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1 - s), \quad s \in \mathbb{C} \]

Conjecture

The non-trivial zeros of the Riemann zeta function \( \zeta(s) \) lie on the line

\[ \text{Re}(s) = \frac{1}{2}. \]
Riemann Hypothesis (RH)

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Equivalent statements of RH

Let

$$\pi(X) = \#\{\text{primes numbers } p \leq X\}.$$

Then

$$RH \iff \left| \pi(X) - \int_2^X \frac{dt}{\log t} \right| = O(X^{1/2+\varepsilon}).$$

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$$\theta(X) = \sum_{p < X} \log p.$$

Then

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RH for a curve $C$ over a finite field $\mathbb{F}_q$

Theorem (Weil)

$$|\#C(\mathbb{F}_{q^n}) - (1 + q^n)| \leq 2 g_C \sqrt{q^n}.$$ 

Remark

To compare with the case for $\mathbb{Q}$:

$$RH \iff |\theta(X) - X| = O(X^{1/2+\epsilon}).$$
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Example

To an elliptic curve over $\mathbb{Q}$

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}$$

Hasse–Weil associates an L-function

$$L(s, E) = \prod_{p: \text{good}} \frac{1}{1 - a_p p^{-s} + p^{1-2s}}$$

where, for good $p$

$$a_p = p + 1 - E(\mathbb{F}_p).$$
There are many other L-functions, e.g., those attached automorphic representations on $\text{GL}(N)$.

**Conjecture**

*Nontrivial zeros of all automorphic L-functions lie on the line*

\[
\text{Re}(s) = \frac{1}{2}.
\]
1. Riemann hypothesis

2. Positivity of L-functions

3. Completion of square

4. Positivity on surfaces
A corollary to Riemann Hypothesis

Suppose that an $L$-function has the following properties

- $L(s)$ is an entire function.
- $L(s) \in \mathbb{R}$ if $s$ is real.
- $L(s) > 0$ as $s \in \mathbb{R}$ and $s \to \infty$.

We have

$$\text{GRH} \implies L(1/2) \geq 0,$$

or more generally, the first non-zero coefficient (i.e., the leading term) in the Taylor expansion is positive.
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Superpositivity: non-leading terms

Lemma (Stark–Zagier (1980), Yun–Zhang)
Let $\pi$ be a **self-dual** cuspidal automorphic representation of $\text{GL}_n$. Normalize its functional equation such that

$$L(s, \pi) = \pm L(1 - s, \pi).$$

Then

$$\text{GRH} \iff L^{(r)}(1/2, \pi) \geq 0, \text{ for all } r \geq 0.$$

Here

$$L(s, \pi) = \sum_{r=0}^{\infty} L^{(r)}(1/2, \pi) \frac{(s - 1/2)^r}{r!}.$$
The idea of proof

Hadamard product expansion (and the functional equation and the self-duality)

\[ L(s + 1/2) = c \cdot s^r \prod_{\rho} \left(1 - \frac{s^2}{\rho^2}\right), \]

- the product runs over all the zeros \( \frac{1}{2} \pm \rho \) of \( L(s) \) such that \( \rho \neq 0 \),
- \( r = \text{ord}_{s=1/2}L(s) \), and \( c > 0 \) is the leading Taylor coefficient.

Now note that

\[ GRH \iff \text{Re}(\rho) = 0. \]
Super-postivity of L-functions

- Super-postivity does not imply GRH. But it implies the non-existence of Landau–Siegel zero.

- Known for Riemann zeta function (Polya, 1927). Sarnak introduced a notion of “positive definite" for L-functions. If an L-function is positive definite then it is “super-positive". Not known if there are infinitely many positive definite L-functions.

- Goldfeld–Huang: there are infinitely many “super-positive" automorphic L-functions for GL(2).
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The super-positivity suggests us
to express $L^{(r)}(1/2, \pi)$ in terms of some “squared quantity”.

We explain two such examples

- (Gross–Zagier, Yuan–Zhang–Zhang) The first derivative

$$L'(1/2, \pi) \geq 0$$

if $\pi$ appears in the cohomology of Shimura curve over a (totally real) number field $F$.

- (Yun–Zhang) “Higher Gross–Zagier formula” over function fields.
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Theorem

Let $E$ be an elliptic curve over $\mathbb{Q}$. There is a point $P \in E(\mathbb{Q})$ such that

$$L'(1, E) = c \cdot \langle P, P \rangle,$$

where the RHS is the Néron–Tate height pairing

$$\langle \cdot, \cdot \rangle : E(\mathbb{Q}) \times E(\mathbb{Q}) \to \mathbb{R}$$

and $c$ is a positive number.

The point $P$ in the above formula is the so-called Heegner point. The Néron–Tate height pairing is known to be positive definite. Hence

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The modular curve $X_0(N)$ is moduli space classifying elliptic curves with auxiliary structure:

$$
\begin{align*}
X_0(N) & \to E \\
\downarrow & \downarrow \\
\text{Spec } \mathbb{Q} & \to \text{Spec } \mathbb{Q}
\end{align*}
$$

The Heegner points are represented by those special elliptic curves with complex multiplication.
Drinfeld Shtukas

Now fix \( k = \mathbb{F}_q \), and \( X/k \) a smooth geometrically connected curve. We consider the moduli stack of Drinfeld Shtukas of rank \( n \). For a \( k \)-scheme \( S \), we have

\[
\text{Sht}_{\text{GL}_n, X}^r(S) = \begin{cases} 
\text{vector bundles } \mathcal{E} \text{ of rank } n \text{ on } X \times S \\
\text{with minimal modification } \mathcal{E} \to (\text{id} \times \text{Frob}_S)^* \mathcal{E} \\
\text{at } r\text{-marked points } x_i: S \to X, 1 \leq i \leq r
\end{cases}
\]

We have

\[
\text{Sht}_{\text{GL}_n, X}^r \downarrow \\
X^r = X \times \text{Spec} k \cdots \times \text{Spec} k \underbrace{X}_{\text{r times}}
\]
Theorem (Yun–Zhang)

Fix \( r \in \mathbb{Z}_{\geq 0} \). Let \( E \) be a semistable elliptic curve over \( k(X) \). Then there is an algebraic cycle (the Heegner–Drinfeld cycle) on \( \text{Sht}^r_{\text{PGL}_2, X} \) such that the \( E \)-isotypic component of the cycle class \( Z_{r,E} \) satisfies

\[
L^{(r)}(1, E) = c \cdot \left( Z_{r,E}, Z_{r,E} \right),
\]

where \( (\cdot, \cdot) \) is the intersection pairing.

The Heegner–Drinfeld cycle is defined analogous to Heegner point on modular curves, by imposing “complex multiplication": those vector bundles coming from a double covering of the curve \( X \).
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Comparison with the number field case

In the number field case, the analogous spaces only exist when $r \leq 1$.

1. When $r = 0$, this is the double-coset space

$$G(F) \backslash (G(\mathbb{A})/K).$$

2. When $r = 1$, the analogous space is Shimura variety

$$X^r = X \times_{\text{Spec}k} \cdots \times_{\text{Spec}k} X$$

$r$ times.
In the function field case, we need not restrict ourselves to the leading coefficient in the Taylor expansion of the $L$-functions.

**Question**

*In the number field case, should there be any geometric interpretation of the non-leading coefficients, for example, $L^{(r)}(1, E)$ when $E$ is an elliptic curve over $\mathbb{Q}$?*

Recall that the conjecture of Birch and Swinnerton-Dyer gives a geometric interpretation of the leading term

$$L^{(r)}(1, E) = c \cdot \text{Reg}_E \cdot \text{III}_E.$$
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Intersection pairing on an algebraic surface

\( S: \) smooth projective surface over a field \( k \).
\( \text{Div}(S): \) free abelian group of divisors on \( S \).
There is an intersection pairing

\[
\text{Div}(S) \times \text{Div}(S) \to \mathbb{Z}
\]
\[
(C, D) \mapsto C \cdot D
\]
Hodge index theorem for a surface

**Theorem**

*Let $S$ be a surface over a field $k$. If $H$ is an ample divisor, and $D \cdot H = 0$, then*

$$D \cdot D \leq 0.$$  

NS$(S) = \text{Div}(S)$ modulo numerical equivalence. Then the index of the intersection matrices of a basis of NS$(S)$ is

$$ (+, -, -, -, \cdots ).$$
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Weil’s proof of RH for curves

Consider a curve $X/\mathbb{F}_q$, and the surface

$$S = X \times_{\text{Spec} \mathbb{F}_q} X$$

Compute the intersection matrix of 4 divisors

$$pt \times X, \quad X \times pt, \quad \Delta, \quad F$$

$F$ is the graph of the Frobenius

$$\text{Frob}_q : X \rightarrow X.$$
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$$\text{Frob}_q : X \to X.$$
Denote $N = X(\overline{\mathbb{F}}_q)$. The intersection matrix

$$T = \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & q \\
1 & 1 & 2 - 2g & N \\
1 & q & N & q(2 - 2g)
\end{pmatrix}$$

$$H = pt \times X + X \times pt \quad \text{ample}$$

$$\implies \det(T) = (N - (1 + q))^2 - 4qg^2 \leq 0$$

$$\implies |N - (1 + q)| \leq 2g\sqrt{q}.$$
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An arithmetic surface $\overline{X}$ is the data of a relative curve $\mathcal{X} \rightarrow \text{Spec}\mathbb{Z}$ with a metric on the Riemann surface $X(\mathbb{C})$.

Arakelov defined an intersection pairing on an arithmetic surface.
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\begin{array}{ccc}
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Theorem (Faltings, Hriljac)

Let $\overline{X}$ be an arithmetic surface.

If $\overline{H}$ is an ample divisor, and $\overline{D} \cdot \overline{H} = 0$, then

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Remark

- This positivity together with Gross–Zagier formula implies $L'(1, E) \geq 0$. (in addition to RH over finite fields)
- Comparison the proof of $L'(1, E) \geq 0$ with the proof of RH for curve over a finite field. The geometric ingredients in them seem to be the best evidence to RH.
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Yuan's proof of Hodge index for arithmetic surfaces

Yuan: an arithmetic line bundle $\mathcal{L} \mapsto$ a convex body in $\mathbb{R}^2$.

**Lemma (Brunn–Minkowski)**

Let $A, B$ be two compact subsets of $\mathbb{R}^n$, and let $A + B$ denote the Minkowski sum

$$A + B = \{a + b : a \in A, b \in B\} \subset \mathbb{R}^n.$$ 

Then

$$\text{vol}(A + B)^{1/n} \geq \text{vol}(A)^{1/n} + \text{vol}(B)^{1/n}.$$
Surfaces

- The first kind is a surface over a field $k$, e.g. $C \times C$ for a curve $C$ over $k$.
- The second kind is arithmetic surface: its base is an arithmetic curve $\text{Spec} \mathbb{Z}$ and its fiber are curves over fields.
- The third kind is unknown: "$\text{Spec} \mathbb{Z} \times_{\text{Spec} F_1} \text{Spec} \mathbb{Z}$"? It should be a fibration with its base an arithmetic curve $\text{Spec} \mathbb{Z}$ and with fibers also being arithmetic curves.
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The third example: ABC and Landau–Seigel zeros

Definition

A Landau–Siegel zero is a zero $\beta$ of $L(s, \chi_d)$ (for the quadratic character $\chi_d$ associated to $\mathbb{Q}[^{\sqrt{d}}]$) lying in

$$[1 - c/ \log |d|, 1]$$

for a small $c > 0$.

Theorem (Granville–Stark)

A uniform (over number fields) version of ABC conjecture implies that there are no Siegel zeros for $L(s, \chi_{-d})$ with $-d < 0$. 
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The key to the theorem of Granville–Stark is Kronecker limit formula for an imaginary quadratic field $K = \mathbb{Q}[\sqrt{-d}]$. This formula relates the Faltings height of an elliptic curve $E_d$ with complex multiplication by $O_K$ to $L$-function

$$h_{\text{Fal}}(E_d) = -\frac{L'(0, \chi_{-d})}{L(0, \chi_{-d})} - \frac{1}{2} \log |d|.$$ 

Colmez conjecture generalizes the identity to CM abelian varieties. An averaged version is recently proved by Yuan–S. Zhang and by Andreatta–Goren–Howard–Pera.
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