PROBLEM SESSION
50 YEARS OF NUMBER THEORY AND RANDOM MATRIX THEORY

(1) Michael Rubinstein. The following random matrix integral arose in the work [10] in connection with the k-fold divisor function in short intervals:

$$\int_{\mathrm{U}(N)} \det(I - xU)^k \det(I - U^*)^k \, dU = \sum_{m=0}^{kN} I_k(m, N) x^m.$$

My former Master's student Andy showed in Theorem 4 of his thesis that this is equal to

$$\frac{c_{N,k}}{(1-x)^{k^2}} \underbrace{\det\left(\frac{1 - x^{N+i+j-1}}{N+i+j-1}\right)_{i,j=1}^{k}}_{=:G_{k,N}(x)}$$

with

$$c_{N,k} = \prod_{j=1}^{k} \frac{(N+k-j-1)!}{(j-1)!^2 \, (N+j-1)!}.$$

I have observed, experimentally, that the function $F := xG'/G$ seems to satisfy the following differential equation:

$$\begin{aligned} x^2(x-1)^2 F''' + x(5x-1)(x-1)F'' + 6x(x-1)^2(F')^2 + 4(x-1)(x+1)FF' \\ + ((-4k^2 - 4Nk - N^2 + 4)x^2 + (4k^2 + 4Nk + 2N^2 - 2)x - N^2)F' \\ + 2F^2 - (2k^2 + 2Nk)F = 0 \end{aligned}$$

I would like a proof that F does indeed satisfy the above equation, and a point of view that explains why.

(2) Jared Lichtman. Show the that following sequence of integrals converges to $e^{-\gamma}$:

$$\int_0^1 \frac{dx}{1+x}, \quad \int_{[0,1]^2} \frac{dx\,dy}{1+x(1+y)}, \quad \int_{[0,1]^3} \frac{dx\,dy\,dz}{1+x(1+y(1+z))}, \ldots$$

This arose while thinking about numbers with k prime factors, $k \to \infty$. This is related to the de Brujin/Dickman function. Hoping for some nice direct proof, which would then generalize to related sequences of integrals. So far, only have a circuitous route to showing this using results on the distribution of numbers with k prime factors. Checks out numerically.

(3) Matthew Young. Come up with a recipe for conjecturing the size of the norm in the large sieve inequality for a family of automorphic forms:

$$\Delta(\mathcal{F}, N) = \max_{\|a\|=1} \sum_{f \in \mathcal{F}} \left| \sum_{n \le N} a_n \lambda_f(n) \right|^2.$$

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Here $\mathcal{F}$ is a "primitive" family of automorphic forms. Avoid "biased sets." Optimistic bound is $N + |\mathcal{F}|$, maybe up to $(N|\mathcal{F}|)^{\varepsilon}$. Maybe don't get that in general, but would be nice at least to have a recipe saying how big this thing is.

- Henryk Iwaniec: A comment. There are conjectures of this type for Dirichlet characters where the families are not complete. Work of Bombieri, Montgomery. Works of Heath-Brown [9] and Dunn-Radziwiłł [2] on Patterson's conjecture.
- Maybe nice to have more examples?

(4) Brad Rodgers. A problem more in random matrix theory and analysis than number theory. Let ω be uniformly distributed on the unit circle S^1. Define a 2×2 matrix $g(\omega)$ by

$$g(\omega) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \omega \\ 1 & -\omega \end{pmatrix} \in \mathrm{U}(2).$$

Consider a sequence of integers n_k which is lacunary in the sense that $n_{k+1}/n_k \geq \lambda > 1$. Consider the sequence of matrix products

$$g(\omega^{n_k})g(\omega^{n_{k-1}}) \cdots g(\omega^{n_1}).$$

The conjecture is that this random variable tends to the Haar measure on $\mathrm{U}(2)$. Currently known in only two special cases, but not in general:

- For $n_k = \lambda^k$ via Brad's work on Rudin-Shapiro polynomials [12]
- For $n_{k+1}/n_k \to \infty$ (master's thesis under preparation).

(5) Aled Walker. Consider Montgomery's Pair Correlation function [11]

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T\right)^{-1} \sum_{0<\gamma,\gamma' \leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma')$$

where $w(u) = 4/(4 + u^2)$ and γ, γ' are ordinates of the zeros of ζ.

Prove that there is an explicit $\alpha > 3/2$ for which $F(\alpha, T) \geq c > 0$ for all $t \geq T_0$, with c and T_0 explicit. Feel free to assume GRH. This is motivated by problems concerning the von Mangoldt function.

A result due to Goldston, Gonek, Özlük and Snyder [5] falls just short of that, it shows that under GRH we have $F(\alpha, T) \geq 3/2 - \alpha$ for $\alpha \in [1, 3/2]$.

(6) Jon Keating. A challenge or problem directed at those who specialize in numerical computations. Compute a negative moment of $\int_0^T |\zeta(1/2 + \alpha + it)|^{-2k} \, dt$ that distinguishes between the conjectures Alexandra Florea stated in her lectures this morning: one coming from RMT while the other was made by Gonek [8].

- Brian Conrey: if α gets too small, then it's just a few individual zeros that control everything.
- Zeev Rudnick. To investigate these negative moments in the function field setting, even in the large-q limit, one would need to think a little bit, because functions are not continuous; inverse characteristic functions have singularities on the unit circle. (Ideally, with fixed q?).

(7) Zeev Rudnick. There's a lemma of Chebyshev that says that the least common multiple of the first N integers has logarithm given by the Chebyshev ψ function at N, which is asymptotic to N by the Prime Number Theorem:

$$\log \mathrm{lcm}\{1, \ldots, N\} = \psi(N) \sim N.$$

In 2011, Javier Cilleruelo raised a variant: take a fixed irreducible polynomial f with integral coefficients, and try to estimate

$$\log \operatorname{lcm}\{f(1), \ldots, f(N)\}.$$

He proved that for quadratic f this is asymptotic to $N \log N$ [1]. He conjectured that this is asymptotic to $(\deg f - 1)N \log N$ for $\deg f \geq 2$. Nothing known for degrees higher than 2. Prove his conjecture in new cases.

(8) Dorian Goldfeld. Has anyone made any conjecture on moments of the Selberg zeta function (for which the analogue of the Riemann hypothesis is known)? Say for $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Answer probably sensitive to issues of arithmeticity (Poisson vs. GOE). Maybe a central limit theorem for $\log\left|Z(\frac{1}{2} + it)\right|$.
   - Zeev Rudnick: maybe fun to do the same but, rather than averaging over t, to average over the Riemann surface. How about gaps between eigenvalues on a Riemann surface. Arbitrarily small gaps? Arbitrarily long gaps? Don't know whether they exist.
   - Matthew Young: maybe don't even know that there are infinitely many simple zeros? For $\mathrm{SL}_2(\mathbb{Z})$, the multiplicity could hypothetically be huge: we only know multiplicity one, but don't know that the eigenvalues are distinct.

(9) Sieg Baluyot. Analogue in RMT for twisting by $(n/m)^{it}$? How about for function fields?

(10) Jeff Lagarias. One of the mysterious questions about the Riemann zeta function, as compared with the Selberg zeta function, is: why is the Riemann zeta function entire of order one, while the Selberg zeta function is of order two? Let's give a version of this problem. Consider the following Dirichlet series:

$$f_b(s) := \sum_{n=1}^{\infty} d_b(n) n^{-s}, \quad d_b(n) = \text{ sum of the digits of } n \text{ in base } b.$$

If $b \geq 2$, then it is known that this function continues meromorphically to the whole plane with simple poles that are arranged in a left half-plane at elements of a two-dimensional half-lattice with periods 1 and $\frac{2\pi i}{\log b}$ (see the paper by his student Everlove [4]). Thus the poles form a two-dimensional half-lattice. Prove that this function is of order 2 (it must be ≥ 2 because it has too many poles). On the other hand, if $b = 1$, then it's a shift of ζ, hence of order 1.

(11) Dan Goldston. Finding the most commonly occurring gap between consecutive primes up to x is sometimes called the "jumping champion problem" (Conway). Say $x = 7$. Then we have $3 - 2 = 1, 5 - 3 = 2, 7 - 5 = 2$, so the jumping champion is 2. We ask another question. Erdös–Straus [3] in 1980 proved that the jumping champions actually have to go to ∞, but that assumes a Hardy–Littlewood prime pair conjecture. If you assume more conjectures, you can say more. Would be nice to prove *anything* without any condition, beyond trivialities like proving that 1 is not a jumping champion. Can't prove anything, period; can't even numerically discover anything. Worked on the problem of trying to find the biggest loser, i.e., some number that will never be a jumping champion, like 2 or powers of 2. Couldn't do that. Can't prove that any given number will never be a

jumping champion. Maybe can prove that there *exists* some even number that is not eventually a jumping champion (i.e., a "loser")? Don't need to specify it; just show that it exists.

Problem: literally prove anything.

(See works with Ledoan [6, 7].)

References

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