Integral Closure, Multiplier Ideals and Cores

The American Institute of Mathematics

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Corrections and new material are welcomed and can be sent to `workshops@aimath.org`

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A.1 Crispin Quinonez, Veronica

I am primarily interested in minimal reductions and cores. In my doctoral thesis I show one way to find a minimal reduction of a monomial ideal in two variables over a field. I am currently working on extending the result to monomial ideals in three variables. I would like to learn more about cores, especially the recent results and ideas by C. Pollini, B. Ulrich and M. A. Vitulli about cores of monomial ideals.

A.2 Enescu, Florian

The workshop is devoted to a long list of important questions in commutative algebra centered around the concept of integral closure. As such, I will list the problems that I am most interested in working on as a participant of the workshop. This list is most likely incomplete, as I am very much interested in learning of new connections and problems during the days of the meeting.

I am interested in studying the issues surrounding the computation of integral closure and the core of ideals. Recently papers by Corso-Polini-Ulrich, Polini-Ulrich, Huneke-Trung, Hyry-Smith have presented formulae for cores of ideals in special cases and raised a number of interesting questions. Among them, I would like to look at the following basic question stated in a paper by Huneke-Trung: Let (R, m) be the localization of a standard graded algebra A over an infinite field at its maximal graded ideal. Is it true that $core(m) = core(A)R$? I am also interested in connections between geometry and algebra via the notion of core, a study initiated by Hyry-Smith. In particular I am interested in a special case of the Kawamata conjecture for section rings S with Gorenstein rational singularities, namely whether $gradedcore(I) = adj(I^d)$, where d is the dimension of S and $I = S_{\geq N}$, $N \gg 0$ and $adj(-)$ stands for the adjoint ideal.

On problems related to my own research, I specify that I am interested in studying the family of Noetherian filtrations of ideals in (polynomial) rings. Among the questions that interest me here is the following one: can filtrations be organized in a “nice” metric space? It is known that Rees has introduced the notion of an integral closure of a filtration. Can one extend the Briançon-Skoda theorem for ideals to a version involving filtrations of ideals? A possible approach here is via the notion of asymptotic multiplier ideal of graded systems of ideals as defined in a paper by Ein, Lazarsfeld, Smith and later in a book of Lazarsfeld.

I would like to close here with a more concrete problem tied to integral closure: Let A be a commutative Noetherian ring with identity and I, J ideals in A with $J \subseteq \sqrt{I}$. Also, assume that the ideal I is not nilpotent and $\bigcap_k I^k = (0)$. Then for each positive integer m one can define $v(m) = v_I(J, m)$ to be the largest integer n such that $J^m \subseteq I^n$. We would like to answer the question of whether this sequence increases eventually in a periodic way; that is, whether or not there exists a positive integer t such that $v(m+t) - v(m+t-1) = v(m) - v(m-1)$ for $m \gg 0$, or equivalently, $v(m+t) - v(m) = constant$, for $m \gg 0$. Ciupercă-Enescu-Spiroff obtained some affirmative partial results (part of work that is currently in submission):

Theorem *Let I, J be ideals in a Noetherian local ring A such that $J \subseteq \sqrt{I}$, the ideals I, J are not nilpotent, and $\bigcap_k I^k = (0)$. Assume that J is principal and the ring $\mathcal{B} = \bigoplus_{m,n} J^m \cap I^n$ is Noetherian. Then there exists a positive integer t such that $v(m+t) = v(m) + v(t)$ for all $m \geq t$.*

Can this result be extended further?

A.3 Gaffney, Terence

My interest in the theory of the integral closure of modules began with the possibility of using this theory to obtain algebraic control of equisingularity conditions such as the Whitney conditions, Thom's A_f condition and the relative form of the Whitney conditions, the W_f condition.

Recently, the multiplicity-polar theorem (described below) offers an opportunity to use the multiplicity of modules, and more generally pairs of modules, to solve enumeration problems.

The work on equisingularity was inspired by work of Teissier ([T-1]), who used the theory of the integral closure of ideals to study the equisingularity of families of hypersurfaces with isolated singularities.

Typically, you have a complex analytic set which is the total space X of the family, and the parameter space Y is embedded as a smooth subspace. The members of the family, denoted X_y, are the fibers of the projection to Y. You want to understand how the smooth points of X meet Y. For example, Whitney's condition A holds along Y if every limit of tangent hyperplanes to the smooth part of X at a point of Y contains the tangent plane to Y at the point. If the singularity of X at $y \in Y$ is sufficiently different from nearby points, then Whitney A will fail at y, and the family is not Whitney A equisingular.

Since these conditions describe how smooth subspaces (strata) of a complex analytic space meet, they are also called stratification conditions.

Modules enter into equisingularity theory as follows. Given a family of spaces, a stratification condition is expressed as an analytic inequality. This inequality often controls geometry at the infinitesimal level; for example, controlling the distance between a sequence of tangent hyperplanes and some fixed plane. This inequality in turn is expressed as an integral closure condition on some module associated to the infinitesimal geometry of the total space of the family. At this level, the theory applies quite generally. (Cf. [**?**], [**?**].)

In the next step, you want to associate invariants to the modules of the members of the family, and you want these invariants to be independent of parameter if and only if the stratification condition holds. For example, for the Whitney conditions, if $X \subset \mathbf{C}^{n+k}$ is a k parameter family of complete intersections with isolated singularities (ICIS), $Y^k = 0 \times \mathbf{C}^k$ with members $(X^d_y, 0)$, then the module we associate to X_y is defined as follows: take a map germ $f_y : \mathbf{C}^n, 0 \to \mathbf{C}^{n-d,0}$ which defines the germ $(X^d_y, 0)$, and multiply the module generated by the partial derivatives of f, denoted $JM(f)$, by the maximal ideal m_y of $\mathcal{O}_{X^d_y,0}$. The module depends on f but its multiplicity doesn't. The theorem is that the pair $(X - Y, Y)$ satisfies the Whitney conditions if and only the multiplicity of $m_y JM(f_y)$ is independent of y ([**?**],[GK2]).The main tool in proving sufficiency is the principle of specialization of integral dependence proved in [**?**].

However, the multiplicity of the module $JM(f_y)$ is only well defined for ICIS singularities, for the module $JM(f_y)$ has finite colength inside the free module that contains it if and only if $f^{-1}(0)$ is an ICIS. In the case of hypersurfaces with non-isolated singularities, one can get around this by using the Segre cycles associated to the ideal $J(f)$ generated by the partial derivatives of the defining equations. (Cf. [G-G].)

Attempting to do this for modules fails to provide invariants which are necessary for the equisingularity conditions to hold. Working with strata of varying dimension seems too difficult for this approach. (An example showing what can go wrong can be found in [**?**]) Instead, we use pairs of modules which allow us to work one stratum at a time. For example, to extend $JM(f)$ to the case where X^d is an equidimensional complex analytic set, with a 1-dimensional singular set, let $H_0(JM(f))$ be those elements of the free module containing $JM(f)$ which are in the integral closure of $JM(f)$ off the singular set; let $H_1(JM(f))$ be the elements of $H_0(JM(f))$ in $JM(f)$ off the origin. Then the multiplicity of the pairs $(JM(f), H_1(JM(f), 0))$, and $(H_1(JM(f))|_H, H_0(JM(f))|_H, z)$ are well defined, where H is a hyperplane transverse to the singular set at a generic point z, and $M|_H$ denotes the module obtained by restriction to H, and these multiplicities are useful in getting necessary and sufficient conditions for equisingularity. (For further description and application of the H_i see [**?**].) Understanding the meaning and method of computing the submodules $H_i(JM(f))$ and the multiplicity of their pairs is something I would like to do.

The multiplicity polar theorem coupled with an appropriate versal deformation theory is a useful tool in understanding the geometry of the multiplicity of the pair. Given a module M of generic rank e which is a submodule of a free $\mathcal{O}_X^{d+k}$ module, we define the polar variety of codimension d by choosing $d+e-1$ general elements and considering the points where the submodule they generate has less than rank e, but the rank of M is still e. Given a family of pairs of modules (M_y, N_y) induced by the restriction of modules (M, N) to the fibers of X over Y, suppose that the set of points where the integral closure of M is not equal to the integral closure of N is finite over Y. Then, roughly speaking, the multiplicity polar theorem says that the change in multiplicity of the pair (M_y, N_y) from the special fiber to the generic fiber is equal to the difference in degree over Y of the polar variety of M of codimension d and that of N at $(0, y)$ in the special fiber. (For a careful statement of the theorem and a proof in the ideal case see [**?**]; the proof in the module case will appear in [**?**].)

As a sample application it is possible to prove:

Theorem ([**?**] Given M a submodule of $\mathcal{O}_{X,x}^p$ of finite colength, X^d equidimensional, choose $d+p-1$ elements which generate a reduction K of M. Denote the matrix whose columns are the $d+p-1$ elements by $[K]$; $[K]$ induces a section of $(\mathbf{C}^{d+p-1}, \mathbf{C}^p)$ which is a trivial bundle over X. Stratify $(\mathbf{C}^{d+p-1}, \mathbf{C}^p)$ by rank. Let $[\epsilon]$ denote a $p \times (d+p-1)$ matrix, whose entries are small, generic constants. Then, on a suitable neighborhood U of x the section of $(\mathbf{C}^{d+p-1}, \mathbf{C}^p)$ induced from $[K]+[\epsilon]$ has at most kernel rank 1, is transverse to the rank stratification, and the number of points where the kernel rank is 1 is $e(M)$.

This gives a geometric interpretation of the Buchsbaum-Rim multiplicity. As a corollary it follows that if $\sigma : E \to F$ is a homomorphism of vector bundles where the rank of E is e and the rank of F is f, $e \geq f$, over X^d, and $e - f + 1 = d$, then the contribution to the $f-1$ degeneracy class of σ at point x, is the Buchsbaum-Rim multiplicity of the module generated by the columns of the matrix associated to σ at x.

In the above theorem, the idea is to construct a family of modules, where the generic member is well understood because it is “stable” then use the multiplicity polar theorem to connect the special and generic fibers. This same technique can be used to count the number of zeroes of a 1-form with an isolated zero on a space with an isolated singularity, using a pair of modules constructed from the 1-form, $JM(f)$ and $H_0(JM(f))$ ([**?**]).

In a final example, in studying the equisingularity of hypersurfaces with a one dimensional singular locus which is complete intersection defined by an ideal I, the pair $(J(f), I)$

plays the same role as $J(f)$ does in the case of isolated singularities. (Here we assume $J(f) = I$ except perhaps at the origin.) It is possible to deform f in such a way that the only singularities of X_y are Whitney umbrella points and smooth curves, and the only singularities of f_y off X_y are Morse singularities. Then $e(J(f), I)$ is the colength of $J(f)$ in I and is also the number of Whitney umbrella points plus the number of Morse points appearing in the generic fiber. Again, the multiplicity polar theorem relates the behavior at the special fiber to the generic fiber. (For details see [G-7])

I would be interested in hearing about other possible applications along these lines.

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A.4 Hochster, Melvin

One of my interests is in the interaction of this area with tight closure theory, including the connections between multiplier ideals and test ideals.

Another is the application of variant notions of integral to understanding the notion of continuous closure introduced recently by H. Brenner. Consider an affine algebraic set over the complex numbers. Continuous closure is define in the coordinate ring of this set by expansion and contraction to the ring of all continuous functions on the set. The continuous closure of an ideal is contained in the integral closure. Brenner defines another notion, axes closure, and proves that it contains the continuous closure. Joint work that I have been doing with Neil Epstein suggests that this is true.

A.5 Howald, Jason

My primary interest is in computational algebraic geometry, particularly with monomial ideals. I have done some work with multiplier ideals as well. I hope to learn about the cores of monomial ideals, because although I know there is recent work there, I'm not familiar with it. But computational questions in general are my cup of tea.

A.6 Huneke, Craig

I view integral closure as a critical concept in the study of commutative rings. It arises from a myriad of conditions; my favorite is that a power series with coefficients in a field of characteristic 0 is always integral over the ideal generated by its partial derivatives. This plays a particularly important role in the study of evolutions, and in trying to understand the Eisenbud-Mazur conjecture.

Integral closure also plays a critical role in the theory of tight closure, and there is a close interplay between these two theories. One obvious example is the tight closure version of the Briançon-Skoda theorem. Less obvious is the parallel results which accompany both theories. Another key component of the study of integral closures is their relation to the mutliplicity and Hilbert functions, and this is again mirrored in the tight closure world by the Hilbert-Kunz multiplicity and Hilbert-Kunz function.

Still another important role which integral closure plays is in the combinatorial study of commutative rings. I am particularly interested in a conjecture (I was instructed in this by R. Villareal) which states that if I is the edge ideal of a clutter, then I has the packing property if and only if the Rees algebra of I is normal.

My final interest in the workshop is the fact that together with Irena Swanson, I just finished a book on integral closures.

A.7 Katz, Daniel

Generally speaking I have an interest in the algebraic aspects of topics related to the integral closure of an ideal or the integral closure of a module. There is also a strong geometric connection with these topics too, in various settings. Historically, the algebraic and geometric views have informed and inspired one another. One example of interest to algebraists is the theorem of Rees concerning multiplicities and integral closure. In the 1950s, Rees, inspired by Zariski and others, developed a theory of valuations associated to an ideal as a means to describe the integral closure of the ideal. This lead Rees to his celebrated multiplicity theorem : Let (R, m) be a formally equi-dimensional local ring and consider two m-primary ideals $J \subseteq I$. Then J and I have the same integral closure, i.e., J is a reduction of I, if and only if J and I have the same multiplicity. Many versions of this theorem have evolved since its first incarnation. However, in the last 15 years or so, thanks to the fascinating work of Kleiman-Thorup, Gaffney and his co-workers, and others, it seems that geometric theorems relating algebraic invariants to the integral closure of ideals and modules have out paced the purely algebraic framework for such results. I am interested in discussing this state of affairs with the algebraists attending the workshop. This then leads to a very general question :

Question 1. To what extent can recent results obtained by Kleiman, Gaffney, and others relating integral closure and algebraic invariants (multiplicity, Milnor numbers, etc) be recovered in a purely algebraic framework?

Following the theme of extending Rees's criterion for equality of integral closure, one has the following question discussed in the past by N. Trung and myself.

Question 2. Let (R, m) be a formally equi-dimensional local ring and I, J two ideals with the same radical, but with neither I nor J necessarily contained in the other. Is there a multiplicity test for equality of integral closures of I and J? When I and J are primary for the maximal ideal, mixed multiplicities work, via the work of Teissier. For $J \subseteq I$, but not necessarily m-primary, a satisfactory result was given by Flenner-Manaresi using j-multiplicities. But, I believe, the general case remains open.

Multiplier ideals are also one of the themes of the workshop and I have a strong interest in learning more about such constructions. In particular, mulitplier ideals played a key role in the proof of a surprising result of Ein-Lazarsfeld-Smith : Let R be a regular local ring which is essentially of finite type over the complex numbers. Let P be a prime ideal of codimension c. Then for all n, $P^{(cn)} \subseteq P^n$. This result was also given by Hochster-Huneke for regular local rings containing a field of characteristic $p > 0$ and also generalized by them in all equi-characteristic cases. Little is known about the case where the ring R has mixed characteristic. There is some hope that Lipman's theory of adjoint ideals might replace the multiplier ideal. In any case, here is an question I have discussed in the past with I. Swanson.

Question 3. Let R be a regular local ring of mixed characteristic and $P \subseteq R$ a prime ideal ? Does there exist an integer c' depending only on the codimension of P so that $P^{(c'n)} \subseteq P^n$, for all n?

The following is a question that seeks to find a geometric answer to an algebraic phenomenon.

Question 4. Suppose S is a power series ring (formal or convergent) over the complex numbers and R is a finite integral extension of S. Let M be a finite S-module and let $\mathcal{F}$ be the minimal resolution of M. Using reduction to characteristic p, Hochster and Huneke have shown that the cycles in the complex $\mathcal{F} \otimes_S R$ are integral over the boundaries of the complex. Can one prove this via analytic or geometric means without passing to rings of characteristic p? This result seems so fundamental that in an analytic or geometric context it ought to be true for purely characteristic zero reasons.

Finally, I also have an interest in the fundamental work of Vasconcelos and his co-workers on computing the integral closure of rings and modules, though again in the setting of mixed characteristic. The computational efforts have often involved an iterative approach. The next question seeks an answer 'up front'.

Question 5. Let S be a normal Noetherian domain and $f(X)$ a monic, separable polynomial with coefficients in S. Let ω be a root of $f(X)$ and let R denote the integral closure of $S[\omega]$, so that R is the integral closure of S in the associated separable extension of the quotient field of S. In this setting there exists a height one unmixed ideal $J \subseteq S[\omega]$ such that $J^{-1} = R$. When can J or at least a primary decomposition of J be explicitly computed, say in terms of the coefficients of $f(X)$ and its derivative? The latter question reduces to the case where S is a discrete valuation ring and the question then has particular interest when S is a discrete valuation ring of mixed characteristic.

A.8 Kleiman, Steven

One of my particular interests lies in the theory of families of complete (integrally closed) ideals of finite colength on a smooth algebraic surface. In the Hilbert scheme, the equisingular ideals form a locally closed subset by work of Noble and Villamayor ('97). Its dimension at a point can be computed in terms of the infinitely near multiplicities of the corresponding ideal on the basis of work by Suhstin ('91) and Roe ('01). In characteristic 0, it can be shown that the subset is smooth. However, the proof is round about, and is based on the deformation theory of sandwiched surface singularities developed by Wahl ('75) and Gustavsen ('98). It would be very good to develop a direct proof, and to decide whether or not smoothness holds in positive characteristic. The issue boils down to this: prove that there is no nontrivial equisingular family of infinitely near points parameterized by the dual numbers and equipped with weights satisfying the proximity inequalities such that the induced family of ideals is trivial.

Another one of my interests lies in the application of the theory of integral dependence to equisingularity theory of complex analytic sets, which another participant, Terry Gaffney, describes in his statement.

A.9 Lee, Kyungyong

Recently Rob Lazarsfeld and I find some necessary conditions for integrally closed ideals to be multiplier ideals. It follows that in dimensions three or higher, multiplier ideals are very special among all integrally closed ideals.

Let X be a smooth complex algebraic variety of dimension d, and let $\mathfrak{b} \subseteq \mathcal{O}_X$ be an ideal sheaf. Given a rational or real number $c > 0$ one can construct the *multiplier ideal*

$$\mathcal{J}(\mathfrak{b}^c) = \mathcal{J}(X, b^c) \subseteq \ \mathcal{O}_X$$

of $\mathfrak{b}$ with weighting coefficient c. This is a new ideal on X that measures in a somewhat subtle manner the singularities of functions $f \in \mathfrak{b}$. In recent years, multiplier ideals have found many applications in local and global algebraic geometry.

Because of their importance, there has been some interest in trying to understand how general or special multiplier ideals may be among all ideal sheaves. Multiplier ideals are always integrally closed, but up to now they have not been known to satisfy any other local properties. In fact, Favre–Jonsson and Lipman–Watanabe proved that in dimension $d = 2$, every integrally closed ideal can locally be realized as a multiplier ideal.

Our main result shows that multiplier ideals satisfy some possibly unexpected properties of an algebraic nature. In the following, we work in the local ring $(\mathcal{O}, m)$ of X at a point $x \in X$, and as above $d = \dim X$.

Let $\mathcal{J} = \mathcal{J}(b^c)_x \subseteq \mathcal{O}$ be the germ at x of any multiplier ideal. If $p \geq 1$, then no minimal p^{th} syzygy of $\mathcal{J}$ vanishes modulo $\mathfrak{m}^{d+1-p}$.

The theorem implies that if $d \geq 3$, then many integrally closed ideals cannot arise as multiplier ideals. For example consider $2 \leq m \leq d-1$ functions

$$f_1, \ldots, f_m \in \mathcal{O}$$

vanishing to order $\geq d$ at x. If the f_i are chosen generally, then the complete intersection ideal $\mathcal{I} = (f_1, \ldots, f_m)$ that they generate will be radical, hence integrally closed. On the other hand, the Koszul syzygies among the f_i violate the condition in the above Theorem, and hence $\mathcal{I}$ is not a multiplier ideal.

A.10 Lipman, Joseph

In [adjoints p. 747], a vanishing conjecture is stated for an ideal I in a d-dimensional regular local ring $(R, \mathfrak{m})$. Suppose there is a map $f\colon X \to \operatorname{Spec}(R)$ which factors into a finite sequence of blowups with smooth centers, and is such that $I\mathcal{O}_X$ is invertible. Let E be the closed fiber $f^{-1}\{\mathfrak{m}\}$. The conjecture is that

$$\mathrm{H}^i_E\big(X, (I\mathcal{O}_X^{-1})\big) = 0 \ \text{ for all } \ i < d.$$

This statement implies a number of Briançon-Skoda type commutative-algebra results, involving adjoints, for any such ideal I. The statement is true if $d = 2$; and was proved by Cutkosky [Dale] for R of finite type over a field of characteristic zero (in which case it is closely related to vanishing theorems which appear in the theory of multiplier ideals.) In these two situations, the assumed principalization is known to exist for any ideal.

Generalizing the $d = 2$ case, I think I can show that vanishing holds for those R-ideals which are *finitely supported*, i.e., for which in the above sequence of blowups, all the centers are closed points. (See [complete, p. 213, (1.20), and p. 215, Remark].)

It would be nice to know what happens with more general ideals. In general, one doesn't expect vanishing theorems to carry over to characteristic p; but we are considering only rather special kinds of maps, so maybe...

In this vein, here is another conjecture (that holds when $d = 2$), for a finitely supported ideal I with adjoint $\tilde{I}$. Any integrally closed R-ideal I is determined by its *point basis* $\mathbf{B}(I)$, which is the family of orders of the completed transforms of I at the "infinitely near points." (See [complete. p. 209].) The ideal I is finitely supported iff all but finitely many integers in $\mathbf{B}(I)$ vanish.

Conjecture. *The point basis* $\mathbf{B}(\tilde{I})$ *can be obtained from* $\mathbf{B}(I)$ *by substituting* $\max(r - d + 1, 0)$ *for each entry* r.

This would immediately imply *subadditivity* of the adjoint operation on finitely supported ideals: $\widetilde{IJ} \subset \tilde{I}\tilde{J}$.

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A.11 Nakamura, Yukio

I am interested in the question that when an ideal sheaf in a regular scheme which is proper birational over a two dimensional local domain having only rational singularitis is generated by a global section.

A.12 Ngo, Trung

My interest in the workshop mainly comes from my interest in reductions of ideals. I have studied the relationships between the reduction number and other invariants of local and graded rings for many years. Recently, I gave a constructive characterization of minimal reductions with given reduction numbers. This characterization has helped me to contribute some results on the core of ideals, one of the main theme of the workshop. But there are some questions left on the nature of the core which I would like to understand. I have also tried to obtain a combinatorial characterization of the integral closure of the Rees algebra of a squarefree monomial ideals and on a numerical characterization of ideals with the same integral closure, another theme of the workshop. By participating to the workshop I hope to get new ideas for the solution of these problems.

A.13 Simis, Aron

My main interest in the workshop relates to the integral closure of a finitely generated submodule of a free module of finite rank over a reasonable ring (Noetherian domain, normal, Cohen–Macaulay, polynomials). In particular I would like to undertsnad what is the nature of both the reflexive and integral closure of basic modules such as the Jacobian module and the module of derivations of a complete intersection, and even important constructions over a polynomial ring over a field, such as the differential idealizer module of a polynomial ideal. I hope to sufficiently interact with other participant and convey to them the relevance of some of these questions.

A.14 Smith, Karen

Some Questions about Cores.

When does an ample line bundle on a smooth projective variety have a non-zero global section? This basic question in algebraic geometry turns out to be equivalent to a fundamental problem in commutative algebra regarding the equality of *core* and *graded core* in a particular situation. I'll mention a few specific problems that turned up in joint work with Eero Hyry on Kawamata's Non-Vanishing Conjecture, where these connections are described. My hope is that someone's interest may be sparked to look at our papers [HS1, HS2], where many more commutative algebra questions are raised whose answers would have a major impact in birational algebraic geometry.

In order for core to have an impact on these kinds of geometric questions, commutative algebraists need to explore two issues. One issue is to understand better the *graded core* of a homogeneous ideal in a graded ring, by which we mean the intersection of the *homogeneous* reductions of I. Somewhat surprisingly, this turns out to be much harder, at least with the techniques currently being used, than looking at the core, even for a homogeneous ideal. The other issue is to understand better a particularly nice type of graded ring called a *section ring*— which, from an algebro-geometric perspective, isn't much different from a standard graded ring. For example, an open question whose positive solution would settle an important conjecture in algebraic geometry is this: **if S is a Gorenstein section ring with rational singularities,**[1] **prove that core I equals graded core I, where I is the ideal generated by all elements of degree at least n, for some (equivalently all) sufficiently large n.**

By definition, a section ring is a ring of the form

$$S = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^n)$$

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[1] Those unfamiliar with rational singularities may replace that hypothesis by the hypothesis that S is regular on the punctured spectrum and has negative a-invariant.

where $\mathcal{L}$ is an ample invertible sheaf on a projective variety X. But there are several ways to characterize them purely algebraically. For example, a normal $\mathbb{N}$-graded domain S which is finitely generated over $S_0 = k$ is a section ring if and only if there exist homogeneous non-zero-divisors $x_1, \ldots, x_r$ generating an -primary ideal such that the $\mathbb{Z}$-graded ring S_{x_i} is isomorphic to the ring $[S_{x_i}]_0[t, t^{-1}]$, where t is an indeterminate of degree one and $[S_{x_i}]_0$ is the subring of the localization ring S_{x_i} consisting of degree zero elements. See Section 2 of [HS2] for the proof of this and other characterizations of section rings. **So one general challenge for the commutative algebra community is to better understand section rings.** For example, what theorems about standard graded rings generalize to section rings?

Another interesting question about which very little is known: **Under what conditions on a graded ring is it true that the core of a homogeneous m-primary ideal is equal to its graded core?** Both the core and graded core of I are homogeneous ideals, and of course the core is contained in the graded core in general. Quite generally, the core and graded core are equal for m-primary ideals generated by elements of the same degree, but this is far from true in general. An important case to understand (coming from the geometry) is the ideal I_N generated by elements of degree at least N, for sufficiently large N. It is shown in [HS1, HS2], that if S is a section ring, then core I_N = graded core I_N for all large N if and only S has an element of degree one. Because Kawamata's Conjecture is known in low dimension (using quite elementary methods such as Riemann-Roch), a challenge for algebraists presents itself: **Find a purely algebraic proof of the fact that if S is a Gorenstein rationally singular section ring of dimension at most three, then S has a degree one element, or equivalently, that core equals graded core for the ideals I_N.** Proving this in higher dimension, of course, would be even more exciting, as it is not yet known there. But any new insight into the "core vs graded core" issue would be interesting!

Another interesting aspect of core is this: one should look not only at the graded core of ideals in a ring S, but also at the graded core of I *in some S-module M*, which is to say, the intersection of the submodules JM inside M where J ranges through all the homogeneous reductions of I. The specific M arising from the geometry is the canonical module of S, but it would be interesting to study this issue more generally as well.

There are many more problems about graded core whose solutions would impact on important questions in algebraic geometry. For example, Kawamata's Conjecture is also equivalent to a formula for graded core in terms of *multiplier ideals*[2] analogous to Huneke and Swanson's formula in [HunSw], so one could try to prove graded versions of their formula. It would also be interesting to understand whether there is a simple formula for the graded core of a homogeneous ideal along the lines of the $J^r : I^{r+1}$ formula for core conjectured by Corso-Polini-Ulrich (and proved in [PU], [HT]). Please take a look at [HS1], and especially the last section of [HS2], for many more (and more precise) commutative alegbra questions impacting on birational algebraic geometry.

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A.15 Takagi, Shunsuke

My interest lies in the theory of multiplier ideals and its characteristic p analogue (generalized test ideals). Hara-Yoshida proved that in a $\mathbb{Q}$-Gorenstein normal ring of characteristic $p \gg 0$, the multiplier ideal $\mathcal{J}(\mathfrak{a})$ coincides with the generalized test ideal $\tau(\mathfrak{a})$. Generalized test ideals can be defined even in non-$\mathbb{Q}$-Gorenstein rings. I would like to give a geometric interpretation of generalized test ideals in non-$\mathbb{Q}$-Gorenstein normal rings.

I am also interested in behavior of symbolic powers of ideals (as an application of the theory of multiplier ideals). Swanson proved that if P is a prime ideal of a normal ring R, then there exists an integer $k = k(P)$ depending on P such that $P^{(km)} \subseteq P^m$ for all integers $m \geq 1$. Using the subadditivity formula of multiplier ideals, Ein-Lazarsfeld-Smith proved that if R is a regular affine $\mathbb{C}$-algebra, then one can take the integer $k(P)$ to be the height of P (later, Hochster-Huneke generalized their result to the case where R is an arbitrary regular ring of equal characteristic). Generalizing their result, I would like to give an effective estimate of $k(P)$ when R is a quotient singularity.

A.16 Teitler, Zach

In recent years, multiplier ideals have emerged as useful tools in algebraic geometry and as interesting objects of study in their own right. However, relatively few explicit examples have been computed. Here is a possibly not-complete list:

1) Monomial ideals, by J. Howald; generalized to toric ideals by M. Blickle

2) General hypersurfaces (meaning, general coefficients)

3) Generic determinantal ideals, by A. Johnson (unpublished)

4) Hyperplane arrangements, by M. Mustaţă; generalized to a locally conical divisor along a stratification by M. Saito

5) Most line arrangements in $\mathbb{C}^3$, by Z. Teitler

In order to state questions, it is useful to define some terminology. Let $Z \subset \mathbb{P}^n$ be a non-empty closed subscheme; the goal is to find the multiplier ideals of $I(Z) \subset \mathbb{C}[x_0, \ldots, x_n]$. For $d \geq 0$, let $Z_d \subseteq \mathbb{P}^n$ be defined by the degree-d homogeneous forms in $I(Z)$. Equivalently, Z_d is the base scheme of the linear series $\mathcal{O}(d) \otimes \mathcal{I}_Z$. Call Z_d the dth degree envelope of Z, or d-envelope. We have

$$\mathbb{P}^n = Z_0 \supseteq Z_1 \supseteq \cdots \supseteq Z_D = Z, \quad D \gg 0$$

Consider how many distinct subschemes appear on this list between $\mathbb{P}^n$ and Z (which are "trivial" degree envelopes). The line arrangements studied above are those corresponding to $Z \subset \mathbb{P}^2$ which satisfy one of the following conditions:

1) There is no intermediate degree envelope. (For example, if Z is three non-collinear points, then $Z_1 = \mathbb{P}^2$ and $Z_2 = Z$.)

2) There is exactly one intermediate degree envelope Z_d and it is a smooth curve. (For example, if Z is five general points, then $Z_1 = \mathbb{P}^2$, Z_2 is a smooth conic, and $Z_3 = Z$.)

3) there is exactly one intermediate degree envelope Z_d and it is a reduced set of points. (For example, if Z is eight general points, then $Z_2 = \mathbb{P}^2$. There is a pencil of cubics through Z, so Z_3 is nine points; and $Z_4 = Z$.)

These are the cases in which the multiplier ideals have been computed. There are a few more cases in which multiplier ideals should be relatively easy to compute.

1) Let Z be 5 points in $\mathbb{P}^2$ lying on a singular conic. Find the multiplier ideals of $I(Z)$. (In this case there is one intermediate degree envelope, but it is not smooth.)

2) Let C be a smooth cubic curve in $\mathbb{P}^2$ and let Z be 11 general points on C. Find the multiplier ideals of $I(Z)$. (In this case there are two intermediate degree envelopes: $Z_3 = C$, Z_4 is a set of twelve points on C, and $Z_5 = Z$. The intermediate degree envelopes have different dimensions.)

3) Fix three general lines in $\mathbb{P}^2$. Let Z be the union of 10 general points on the first line, 7 general points on the second line, 4 general points on the third line, and one more general point in $\mathbb{P}^2$. Again, find the multiplier ideals of $I(Z)$. (In this case there are three intermediate degree envelopes, each a union of lines.)

Generalizing slightly, the question arises which exceptional divisors impose restrictions on the multiplier ideals. In computing multiplier ideals for a line arrangement in $\mathbb{C}^3$, the exceptional divisors form essentially a "chain", in the sense that the incidence graph of the irreducible exceptional divisors is a path. In these cases the relevant exceptional divisors are the first and last ones in the path. I suspect that in some of the "relatively easy cases" above, the incidence graph of the resolution should have forks—corresponding for instance to singularities on a degree envelope, or to "switching" from one degree envelope to the next. One may ask whether these "forks" are relevant for the multiplier ideals.

A.17 Vasconcelos, Wolmer

Topic: Integral Closure and Complexity

The expertises and interests of several of the partipants makes for a unique opportunity for tackling open problems here, development of new strategies and in the formulation of new questions. We briefly sketch one area for focused activity.

Let R be a normal, unmixed integral domain and let A be a semistandard graded R-algebra of integral closure B. The estimation of the number of steps that general algorithms must traverse to build B can be viewed as an invariant of A. It is natural to seek to use these ‘invariants’ to study the number and the distribution of the degrees of the generators of B.

In the particular–but already important–case of Rees algebras of ideals and modules, one should expect to derive very direct expressions of these complexities in terms of multiplicity-based invariants of the ideals and modules. That is already the case for primary ideals. There remains to obtain general expression for the degree data of B. What other invariants of A play roles here?

A.18 Yuen, Cornelia

My research is in jet schemes, and I’m certainly interested to see connections between jet scheme and multiplier ideals. I would also like to learn more about test ideals and multiplier ideals. Finally, I’m working on computational questions of multiplicity (of ideals along their minimal primes), and would like to learn any methods and techniques that might apply to the calculation of multiplicities.

A.19 Zarzuela, Santiago

I’m primarily interested on how the notion of integral closure and related problems should be understood in the frame of the theory of modules. On the other side, I would like to understand in a more purely algebraic way the work of T. Gaffney and others. I very much like the construction of Bourbaki ideals and the recent preprint of Hong-Ulrich “Specialization and integral closure” is one good example of what kind of answer I would like to get.