Nonlinear PDEs of mixed type arising in mechanics and geometry

The American Institute of Mathematics

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Corrections and new material are welcomed and can be sent to `workshops@aimath.org`

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A.1 Bae, Myoungjean

One of important problems in mathematical fluid dynamics is the transonic shock problem of the Euler system for inviscid compressible flow. And in order to solve a transonic shock problem, we encounter with solving mixed type partial differential equations with a free boundary.
Main tasks to study transonic shock problems are:

- *to identify the location of transonic shock across which the type of Euler equations is transitioned from hyperbolic to elliptic(for steady state problem)*;
- *to solve the Euler equations behind the transonic shock.*

And always, many other questions follow as well. For example, one may ask the following questions:

1. *Does there exists unique transonic shock solution?*
2. *If so, is the solution stable under small perturbation of data or domain?*

My research is mostly focused on studying the questions above for multidimensional potential flow. In particular, I am working on a transonic shock problem for'non-isentropic' potential flow.'

A.2 Chen, Jun

I work on mixed type equations arising from gas dynamics. My main interest is multi-dimensional Euler equations. The mixed type equations are corresponding to the transonic flows, which means the speed is above sound speed at some place and below sound speed at other place. I study transonic flows in nozzles and past wedges. By study of steady solutions of Euler equations, we can obtain better understanding about Euler system, and possibly solve more challenging problems, such as regular reflections and Mach reflections.

A.3 Chen, Shuxing

In gas dynamics we often meet various problems involving both supersonic flow and subsonic flow. Some problems related to the transmission from supersonic flow to subsonic flow by passing across a shock front has been studied. However, more complicated case is how a subsonic flow transforms to a supersonic flow smoothly. The situation happens in the problems like supersonic flow passing aroud a blunt body or a subsonic flow with near sonic speed passing de Laval nozzle etc. Another situation is a supersonic flow is adjacent to a subsonic flow via a contact wave. This happens in Mach reflection of shock. The problem can be fomulated to a Tricomi problem of a Lavlentiev-Bitsadze type equation. Generally, the location of the line, where the equation changes its type, is unknown. However, even for the case when the line of changing type is known, the related problem is still open for generic case. To study the partial differential equations of mixed type may also need some fudamental works. I also interested in finding fudamental solutions of mixed type equations and understanding some basic properties of the solutions to mixed type equations.

A.4 Clelland, Jeanne

I study PDEs mostly using the theory of exterior differential systems. This theory works best for PDEs of constant type, and for real analytic solutions. Many of the main results are inapplicable without such regularity assumptions, so the theory is of little use for studying equations of mixed type, or for singular solutions.

Recently in joint work with Marek Kossowski and George Wilkens, we have used intermediate PDEs to construct certain types of singular solutions for hyperbolic-parabolic type-changing PDEs. This was my first experience with PDEs of mixed type, and I hope to learn more about effective methods for studying such equations.

A.5 Feldman, Mikhail

I work on steady and self-similar transonic shocks for Euler equations and potential flow. This involves study of free boundary problems for mixed-type equations.

A.6 Fetecau, Razvan

Working on the derivation of a high order modulation theory for the Boussinesq equations, I arrived at a very interesting system of nonlinear PDE's. At first order, the modulation equations consist of a system of conservation laws that may exhibit wave breaking, as well as ellipticity breakdown (or ill-posedness due to loss of hyperbolicity of the system). We observe numerically that higher order corrections fix the ellipticity breakdown, but we lack an analytical understanding of this phenomenon. The modulation theory applies to very interesting meteorological phenomena, such as the formation of a wave-induced horizontal mean flow(wind) as a result of vertical propagation of waves through the stratified atmosphere.

A.7 Han, Qing

Partial differential equations of mixed type arise in many geometric problems, among which is the isometric embedding of surfaces in $\mathbb{R}^3$ with Gauss curvature changing sign. The underlying differential equation for such an isometric embedding is given by the Darboux equation, a fully nonlinear differential equation of Monge-Ampere type. The Gauss curvature K determines the type of this equation, which is elliptic if K is positive, hyperbolic if K is negative and of mixed type if K changes sign. Little is known when K changes sign. In its simplest form, a Monge-Ampere equation in $\mathbb{R}^2$ has the form

$$u_{xx}u_{yy} - u_{xy}^2 = K(x, y),$$

where K is a smooth function. For the local problem, we are interested in whether we can find a solution u in a neighborhood of the origin if K changes sign there. A special case is given when K changes sign across finitely many smooth curves intersecting at the origin. For example, K is given by a product of finitely many linear functions. A global problem is to isometrically embed a metric on torus in $\mathbb{R}^3$. A necessary condition is that the total positive Gauss curvature is not less than 4π.

A.8 Jang, Juhi

Soap bubbles, airplanes, water waves, and many more geometric or daily life objects can be viewed as solutions of PDEs. Indeed, any physical theory can be realized by a PDE. More ambitiously, the universe that characterizes a physical theory is the manifold that identifies the corresponding PDE, for instance the General Relativity. The geometric structures of manifolds (set of solutions to PDEs) is important to study PDEs, and vice versa. This would be just one naive perspective on PDEs and geometry. My one particular interest is the asymptotic relationship between nonlinear wave equations and some geometric objects like minimal surfaces.

Nonlinear PDEs of mixed type often arise in mechanics and other areas, for instance transonic and sonic flows. These concepts are related to Gauss's theorem in the theory of differential geometry. I believe that the better understanding of geometry can give new and more general ideas in PDEs. At this workshop, I expect to learn more about the whole subject, get the sense of it, and find and get involved in some good problems. On the other hand, the energy and the scaling of given PDEs are the most important tools to study them, and I'd like to see how these methods can be adapted and related from geometry viewpoint.

A.9 Jegdic, Katarina

My Ph.D. thesis is on analysis of a spacetime discontinuous Galerkin method for systems of conservation laws. After receiving my degree, I spent two years as a visiting assistant professor at the University of Houston under supervision of Barbara Lee Keyfitz and Sunčica Čanić. Our goal was to study Riemann problems for several systems of conservation laws that model shock reflection. The main idea, following earlier studies in [CKK1, CKK3, CKL, K], was to write these systems in self-similar coodrdinates and to obtain a free boundary problem for the subsonic state and a reflected shock. One of the difficulties of this approach is that the reduced system changes type (from hyperbolic to elliptic or mixed hyperbolic-elliptic). Using the standard theory of one-dimensional hyperbolic conservation laws, the theory of second order elliptic equations with mixed boundary conditions and the fixed point theorems, we showed existence of local solutions to several problems (the unsteady transonic small disturbance equation and the nonlinear wave system) in [JKC1, JKC2]. Currently, we are interested in extending these ideas to the study of Riemann problems for the isentropic and full gas dynamics equations.

I am looking forward to exchanging ideas and learning about recent developments in this area at the workshop.

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A.10 Liu, Tai-Ping

It'd be nice to discuss the scattering wave patterns consisting of self-similar and stationary gas flows. Also flows through the nozzle represent important stability issues.

A.11 Otway, Thomas

1. An obstacle to progress on nonlinear elliptic-hyperbolic problems may be the fact that so much of the linear theory remains inaccessible. The second-order normal forms are either of Tricomi type or Keldysh type, but almost nothing is known about the latter class of equations, although they arise in many areas of physics and geometry. In particular, one would like to have some a priori information on what kind of regularity to expect in solutions to linear second-order elliptic-hyperbolic equations which are not of Tricomi type.

2. There are also problems in relating the results obtained by linearization to properties of the original nonlinear system. The most common linearization map, the Legendre transformation, is likely to be singular on sets of positive measure outside the elliptic region. Related to this is the question of the invertibility of boundary-value problems in the hodograph plane (for two-dimensional problems). One would like sufficient conditions for invertibility for each of the two normal forms.

3. Is there a rigorous elliptic-hyperbolic variational theory corresponding to the recent model of Mars-Senovilla-Vera (arXiv:0710.0820v1 [gr-qc]) for signature change on a brane?

4. A few years ago, Torre showed that the helically reduced wave equation, which is elliptic-hyperbolic, could be represented as a symmetric positive system and thus possessed strong solutions. This is surprising as both the helically reduced wave equation and the theory of symmetric positive operators have been known for a long time. There are existence theorems for quasilinear symmetric positive systems due to Gu and Tso. These methods tend to be most applicable on discs or annuli, which are appropriate to general relativity. Are there multipliers that could transform interesting forms of the Einstein equations into quasilinear symmetric positive form? Are the physically natural boundary conditions admissible in the sense of Friedrichs?

A.12 Slemrod, Marshall

My work in the last few years has centered on the mathematics of steady transonic flow over an airfoil. The flow is assumed to be in two space dimensions and irrotational. This leads to a system of two quasi-linear partial differential equations that exhibit a change of type from elliptic to hyperbolic across an unknown free boundary. With G Q Chen and Dehua Wang I have been able to give an existence theorem extending earlier work of Cathleen Morawetz. As this work was nearing its conclusion we realized that the weak convergence methods we had used for transonic flow may also work for the isometric immersion problem in classical differential geometry. In fact we have had some success and established an existence theorem for surfaces with negative Gauss curvature within a class prescribed initial and boundary conditions. The next step will try to extend our results to mixed problems similar to transonic flow. As is well known the mixed problem in geometry has had some success for local smooth solutions and we hope our methods may shed some light on global weak solutions of the underlying partial differential equations, i.e. the Gauss-Codazzi system.

A.13 Smoller, Joel

I have lately been concerned with the existence and nonlinear dynamic stability of rotating star solutions to the compressible Euler-Poisson equations in 3 spatial dimensions. We have proved some general theorems which we have applied to White Dwarf and "supermassive" stars, stars which are in convective equilibrium and have uniform chemical composition.

I am also continuing my work on the stability of black holes, and related areas such as a rigorous proof of energy extraction from rotating Kerr black holes. This area is connected with differential geometry and could be of relevance to participants.

A.14 Torres, Monica

I am interested in the structure of entropy solutions of degenerated parabolic-hyperbolic equations. In the domain of the solution we find shock waves and free boundaries (where the equation changes type). I would like to approach the study of these equations using the theory of divergence-measure fields.

A.15 Wang, Dehua

I am interested in mixed type problems arising in mechanics, biomechanics, geometry, and other possible areas. I would like to see the latest progress, open problems, and possible new directions of related research.

A.16 Zheng, Yuxi

I will be interested in two-dimensional Riemann problems for conservation laws. There are various cases. Some of them involve changes of types of equations. Some of them have mysterious shock formation. One of the most interesting discoveries in this area is the Riemann quantities and characteristic decompositions. A large number of cases can be solved now.