Geometry and representation theory of tensors

The American Institute of Mathematics

The following compilation of participant contributions is only intended as a lead-in to the AIM workshop "Geometry and representation theory of tensors." This material is not for public distribution.

Corrections and new material are welcomed and can be sent to workshops@aimath.org

Version: Mon Jul 21 19:36:20 2008

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A.1 Bernardi, Alessandra

My interest in this conference is motivated by my mathematical interests that are on Varieties parameterizing Tensors, dimension of their Secant Varieties and Generation of their Ideals.

A.2 Boralevi, Ada

Some of the geometric tools that used to approach the subjects of this AIM workshop are strictly entangled with my PhD research. (In my thesis I use representation theory of quivers to study the cohomology of homogeneous vector bundles on flag manifolds.)

I will defend my thesis -and hopefully graduate- in July, right before flying to Berkeley, where I will attend the related MSRI School that carries the same title as the AIM workshop.

I am truly interested in the possibility of linking the Algebraic Geometry and Representation Theory that I know with subjects that are new for me like computer science and statistics. All in all one could say that I am attending the School and Workshop mainly to learn.

Also, I am moving to Texas A&M University starting from Fall 2008, where I will spend one year as Visiting Assistant Professor, to work with Prof. Landsberg, and I see this workshop as a great starting point

A.3 Buergisser, Peter

I am one of the authors of the monograph "Algebraic Complexity Theory". A greater part of that book is dedicated to bilinear complexity and tensor rank. The main motivation for the development of that theory is the famous, still unsolved problem of the (asymptotic) complexity of matrix multiplication.

Strassen was the first to make the connection of computational complexity to questions of geometry and representation theory. Mulmuley and Sohoni took up and further developed some of these ideas in view of the P-NP question. However, the resulting mathematical problems turn out to be formidable. For further progress along these lines, it seems that a good understanding of the so-called *Kronecker coefficients* is essential. Kronecker coefficients are the multiplicities in the tensor product decomposition of two irreducible representations of the symmetric group S_n. E.g., this gives information about the irreducible constituents of the vanishing ideal of secant varieties to Segre varieties. (For recent results see Landsberg and Manivel.)

Little is known about the Kronecker coefficients: in general we don't even have a combinatorial interpretation of these numbers. Recently, in joint work with my student Ikenmeyer (presented at FPSAC 08), we proved that the computation of Kronecker coefficients is $\#P$-hard, confirming the general belief that this is a difficult problem.

While this is bad news, there is some hope that the *positivity* of Kronecker coefficients could be decidable in polynomial time. (This is a conjecture due due Ketan Mumuley.) The main ground for this conjecture is the surprising fact that the positivity of the related *Littlewood-Richardson coefficients* can be decided in polynomial time (while computing the coefficients is $\#P$-hard). This is a consequence of Knutson and Tao's proof of the saturation conjecture, combined with the fact that linear programming is solvable in polynomial time.

In recent work with my student Ikenmeyer, we designed a combinatorial polynomial time algorithm for deciding the positivity of Littlewood-Richardson coefficients, based on ideas for optimizing flows in networks. We hope that our planned implementation of this algorithm will be beat all known algorithms that actually compute the LR coefficients.

One of my goals in attending this workshop is to share these ideas of geometric complexity and to discuss their potential. Also, I am eager to learn about the new connections of tensor rank to other areas. Finally, I hope that my knowledge of algebraic complexity will be beneficial to the other participants of the workshop.

A.4 Cai, Jin-Yi

I have been working on holographic algorithms, which is a method to use linear algebraic (tensor theoretic) superpositions of perfect matchings to derive some unexpected polynomial time algorithms.

Some specific questions one can address:

1. To understand block-wise symmetric (standard) signatures. These are signatures directly represented by planar perfect matchings, but external connections are groups in groups of k each, and the signature is symmetric in terms of these groupings.

2. Basis collapse theorems. We are able to prove a basis collapse theorem for any two vector basis in dimension 2^k to 2. However it is open what one can do if one were to use 4 by 4 matrices, or 8 by 4 matrices. In other words, suppose we are to use 4 vector basis in dimension 4 or 8 or higher.

3. In more recent work using holographic reductions to prove complexity dichotomy theorems, how does one generalize the framework to symmetric signatures on n inputs, each of which takes values on $1, 2, \ldots, k$, where $k > 2$. (The theory for $k = 2$ is quite well developed.)

4. A concrete problem in this direction of problem 3: Consider a symmetric function of n inputs, each taking value from $1, 2, 3$. (i.e. the case where $k = 3$.) It can be shown that such a function, listed as a truth table in a column vector of dimension 3^n, (but with only $\binom{n+2}{2}$ possible distinct values), can always be expressed as the following:

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}^{\otimes n} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}^{\otimes n} + \cdots + \begin{bmatrix} a_m \\ b_m \\ c_m \end{bmatrix}^{\otimes n},$$

for suitable m and a_i, b_i, c_i. One can show that $m = \binom{n+2}{2}$ suffices. What is the minimum such m and how to derive such an expression in general for the smallest m?

A.5 Comon, Pierre

I have been faced to tensor problems since 1990, and to my knowledge, some of them still remain incompletely solved:

P1. Global optimization of a multimodal rational function often occurs, e.g. in signal processing. General approaches based on Gröbner bases cannot handle them for these orders of degree and number of variables. Can we use the fact that the latter equations are particular, since they are derivatives of the same objective, to devise an efficient algorithm?

P2. Orthogonal diagonalization is one instance of the previous problem, which consists of the maximization of a polynomial in the entries of an orthogonal matrix, of the form $\sum_p |\sum_{ijk\ell} \mathcal{T}_{ijk\ell} Q_{ip} Q_{jp} Q_{kp} Q_{\ell p}|^2$. Applying a sequence of plane rotations, each reaching the absolute maximum, (Jacobi sweeping) has not been yet shown to always converge to the absolute maximum.

P3. Invertible diagonalization. Trying to approximately diagonalize a symmetric tensor by invertible congruent transforms is also relevant.

P4. Approximation of a tensor by another of lower rank. In Engineering, because of additive noise, the tensor rank is almost always equal to the generic rank. However, the noiseless part, which is of interest, has a lower rank. Being able to find the best low-rank approximate means being able to remove the noise effect. The problem is well-posed in a few cases.

P5. An unsymmetric AH theorem. Tensors with a generic rank are the most interesting in Engineering. In the symmetric (complex) case, we know what they are, thanks to Alexander-Hirschowitz theorem. But can one state a similar theorem for unsymmetric tensors, that is, to homogeneous polynomials of degree d in n variables, but of partial degree 1 in every variable? The numerical results we have obtained so far when dimensions are all equal show that the defective cases (exceptions to the ceil rule) are not the same.

P6. Orbits. For small dimensions, e.g. 4 or 5, we have the list of orbits (w.r.t. the group of invertible linear transforms), both symmetric and unsymmetric; this is known for dimensions 2x2x2 and 3x3x3 for instance. Did someone make this list for higher numbers of variables, and higher degrees?

P7. Algebraic decomposition algorithms. When the rank of a tensor is subgeneric, its decomposition into a sum of rank-1 terms is essentially unique. We have developed an algebraic algorithm for computing the decomposition of a symmetric tensor when essentially unique, but algorithms for the non symmetric case are still missing.

P8. Complexity/accuracy trade-off. Simpler algorithms, able to avoid local extrema, may be more attractive in Engineering.

P9. Maximal rank. To my knowledge, the value of the maximal achievable tensor rank is still unknown for general values of orders and dimensions. We only have bounds, sometimes loose.

P10. Real tensors. Decomposing real tensors in the real field is more complicated. In particular, there can be several typical ranks for certain dimensions. We started to look at the value of typical ranks as a function of order and dimensions.

P11. Positive tensors. Tensors with real positive elements are useful in Spectral analysis in general. Their (asymmetric) decomposition should also involve real positive coefficients. Symmetric tensors associated with positive definite quantics are useful in some medical imaging applications.

P12. Topological issues. It has been incompletely proved that the set of tensors of at most rank r is closed only for $r = 1$ or r maximal. Simple examples of sequences of tensors of arbitrary rank $r > 1$ converging to a tensor of rank arbitrarily larger, $r + p$, smaller than or equal to the maximal rank, would be welcome.

P13. Partial symmetry. It happens in Engineering that tensors are symmetric in several modes, and not in others. The decomposition of the Characteristic function, or Indscal in data analysis, are two such examples. Other types of symmetry also occur for complex tensors, which may be Hermitian symmetric in some modes, and complex symmetric in others. BIOME and FOOBI algorithms take into account those symmetries.

P14. Rank deflation procedure. Since the set of rank-1 tensors is closed (determinantal variety), the best rank-1 approximate exists. One could think that, by subtracting the latter best approximation, the tensor rank would decrease by 1, as it is the case for matrices. Of course, this does not happen for tensors.

P15. Symmetric rank. When decomposing a symmetric tensor into a sum of rank-1 terms, one can impose symmetry to each term, or not. We can thus define the *symmetric rank* of a symmetric tensor. Can we prove that symmetric rank and rank are always the same?

P16. The super-generic case. When the rank of a tensor is above the generic rank, the decomposition into rank-1 terms is not essentially unique. It is still useful in some cases to provide one simple representative of the equivalence class of decompositions. I have this only in dimension 2.

P17. Structured decompositions. In some applications, it is relevant to decompose a tensor into sum of terms of given rank. Blind identification of complex mixtures of complex random variables based on the characteristic function involves rank-2 terms for instance. This may be seen as a block-diagonalization. Other applications have been recently pointed out in telecommunications.

P18. Joint decompositions. One way to face the lack of uniqueness of a decomposition is to use another tensor (or several others), which is supposed to share common modes. The choice and availability of this additional tensor depends on the application. Yet, decomposing jointly several tensors can be viewed as decomposing a single tensor of higher order. Hence, the generic rank is increased and the decomposition may become unique. Joint decompositions may also improve the *numerical stability* of the decomposition algorithm: I conjecture that the smaller the rank compared to generic, the more stable the decomposition.

A.6 De Lathauwer, Lieven

My knowledge of algebraic geometry is quite limited, so I will mainly attend the workshop to learn. I'll be happy to discuss about my work on tensor methods for signal processing, data analysis, etc. Two topics that the audience may find interesting are i) Independent Component Analysis, and ii) the recently introduced concept of Block Term Decompositions.

A problem that the audience also may find interesting, is the following. It seems that the study of rank-related issues so far has been limited to either unsymmetric tensors or (fully) symmetric tensors (the latter being tensors that are invariant under arbitrary index permutations). In Independent Component Analysis however, we often encounter tensors that have complex Hermitean-type symmetries. For instance, complex fourth-order cumulants may be defined by

$$c_{ijkl} = E\{x_i x_j^* x_k x_l^*\} - E\{x_i x_j^*\}E\{x_k x_l^*\} - E\{x_i x_k\}E\{x_j^* x_l^*\} - E\{x_i x_l^*\}E\{x_j^* x_k\},$$

with $E\{\cdot\}$ the statistical expectation and $\cdot^*$ the complex conjugate. Obviously, this tensor has the symmetries

$$c_{ijkl} = c_{kjil} = c_{ilkj} = c_{jilk}^*.$$

Let us consider the following decomposition of $\mathcal{C}$:

$$\mathcal{C} = \sum_{r=1}^{R} \pm \mathbf{v}_r \circ \mathbf{v}_r^* \circ \mathbf{v}_r \circ \mathbf{v}_r^*,$$

with $\circ$ the outer product. Does there exist a generic rank for such tensors? If yes, how can it be computed?

A.7 Friedland, Shmuel

I am interested in the following problems.

1. To prove a disprove the conjecture for the generic rank of three tensor of dimension (I, J, K), where $I \le J \le K \le (I-1)(J-1)$. Namely for $(I, J, K) \ne (3, 2p+1, 2p+1)$ the generic rank is the smallest integer greater equal to $I * J * K/(I + J + K - 2)$.

2. To characterize the variety obtained by the closure of all $(4, 4, 4)$ tensors of rank 4. Is it defined by a set of polynomials of degree 9 at most?

3.To find good algorithms to approximate a given 3 tensor by low rank tensors.

A.8 Gour, Gilad

I would hope to examine in the workshop the long standing additivity conjecture that the minimum entropy output of a completely positive trace preserving linear map, as measured using the von Neumann entropy, is additive under taking tensor products. Despite the enormous effort by the most experts in the field during the last 12 years, this important problem remains unsolved. Perhaps one of the reasons for that is that the mathematics involved can be formidable, hence avoided, by many physicists. The conjecture has also been shown by Shor and others to be equivalent to a number of other quantum information conjectures, including the additivity of entanglement of formation, and strong superadditivity of entanglement of formation, and additivity of the Holevo capacity of a quantum channel.

A.9 Kondor, Imre Risi

I am interested in the applications of non-commutative harmonic analysis to machine learning and statistical learning theory. This is a very new field, but it is starting to attract a fair amount of interest in the machine learning comunity. The applications explored so far include the track/target matching problem in multi-object tracking, new ways of constructing invariant features for images, connections to ranking problems and new graph invariants. Further material, including a software library implementing Clausen's FFT for the symmetric group may be found on my web page: http://www.gatsby.ucl.ac.uk/∼risi.

A.10 Landsberg, Joseph

I am an organizer for this workshop, which involves several areas so please forgive the length of this contribution. Recently I have realized how objects I have been studying most of my mathematical life (secant, tangential and dual varieties of G-varieties) have important uses in applications to computer science, statistics and other areas (henceforth all called "areas of applications").

My main goal for this workshop is for all of us to develop means of communicating with each other. Personally I would like to get a precise understanding of what problems in the geometry of tensors are relevant for the areas of application, and to educate myself (just out of scientific curiosity) about these areas of application. As a service both to geometers and researchers in the areas of applications, I hope for us to create a list of important questions about spaces of tensors from the areas of applications. This list would be circulated publicly and distributed to geometers. From my experience already, there will be a wide range of the difficulty of these questions (only loosely correlated with the significance of having an answer to them!), ranging from those that might be answered during the workshop, to those appropriate for a PhD thesis, to those that may serve as motivating problems for years to come. On the other side, I hope that the geometers will be able to help researchers in the areas of applications understand what tools and results are already available to help them in their work.

Here are specific questions, organized by area:

I. Geometry 1. Determine defining equations for the secant varieties (i.e. varietes of tensors/polynomials of bounded border rank) of Segre varieties (variety of rank one tensors) and Veronese varieties (variety of rank one tensors). 2. Determine systematic methods for writing down such equations using contractions and wiring diagrams. 3. Develop techniques for giving a geometric description of the zero set of a given module of polynomials. 4. Compare the stratifications of spaces of tensors/polynomials given by border rank and singularities of dual varieties. (This is related to comparing rank and border rank.)

II. Complexity 1. "Geometrize" Schonhage's "approximate algorithms" for fast matrix multiplication. 2. Understand the geometry behind matchgates/holographic algorithms. 3. Obtain a geometric description of the classes VP and VNP and use such descriptions to compare them. 4. Understand the Mulmuley-Sohoni approach to P vs NP via representation theory.

III. Signal processing 1. Determine the possible discrepancies between rank and border rank of tensors and symmetric tensors (polynomials) 2. (CGLM question): is the rank (resp. border rank) of a polynomial considered as a symmetric tensor the same as its rank (resp. border rank) considered as a tensor? 3. Understand the motivation for these and other questions (e.g. I.1) from independent component analysis.

IV. Quantum information (Remark: this area of application is different from the others in that it has already been well examined by experts in representation theory, if not in geometry.) 1. Determine which classes of tensors correspond to those constructed via "graph states", and their possible uses in geometry and the other areas of application. 2. Compare the geometry of different measures of entanglement 3. Study the "additivity conjecture"

A.11 Oeding, Luke

Suppose X is a G-variety and suppose I is a candidate for the ideal of X. What properties of I coming from the G-module structure, translate to commutative algebra structure? For example, if you have a set of candidate generators for $I(X)$ as a direct sum of irreducible G-modules, what are some purely G-module conditions on I that will ensure that you have generated a prime ideal? How can modern computational techniques be modified to make use of the group structure? I think that these and other similar questions could bring to light useful connections between the area of representation theory and the areas of algebraic geometry and commutative algebra.

A.12 Ottaviani, Giorgio

My interest in the topic of the workshop comes from the study of higher secant varieties in algebraic geometry. The k-secant varieties of the Segre varieties (product of projective spaces) correspond to tensors which have border rank bounded by k, and an open (dense) subvariety parametrizes tensors which have rank bounded by k. In the same way the k-secant varieties of the Veronese varieties correspond to symmetric tensors and the k-secant varieties of the Grassmann varieties correspond to skew-symmetric tensors. There is a lot of classical study on these objects and in many small dimensional cases the equations and the dimension of these varieties are well understood, but their general behaviour is a difficult open problem. The dimension has an expected upper bound, but in some cases, called defective ones, the dimension becomes smaller. One of the easiest examples is provided by 2x2x2x2 tensors. To give a more sophisticated example, consider the cubic polynomials in five variables of rank bounded by seven. They form a hypersurface, and its equation has degree 15, with an interesting feature, it comes from a symmetric 45x45 determinant which is a cube. The most striking positive result is the celebrated Alexander-Hirschowitz theorem, that in the symmetric case classifies all defective cases, which correspond to interesting geometrical situations. The symmetric case corresponds to polynomial interpolation.

In different collaborations, with C. Brambilla and with H. Abo and C. Peterson, I have studied the general and the skew-symmetric case, trying to generalize the prof of Alexander-Hirschowitz theorem, providing an inductive technique which makes possible the computation of the dimension in many cases. For three dimensional tensors this inductive technique turned out to be equivalent to a technique developed in the book Burgisser, Clausen, Shokrollahi, Algebraic Complexity theory. I hope that the workshop will be useful to develop a common language and to link the theory with the applications.

A.13 Roy, Aidan

I am interested in various aspects of representation theory and geometry as it applies of quantum information theory. In particular:

1) the additivity conjecture for entanglement of quantum channels;

2) the geometry of optimal measurements for quantum state tomography;

3) hidden variable models of quantum dynamics.

In this workshop, I would like to further my understanding of how algebraic geometric tools can be applied to these problems.

A.14 Valiant, Leslie

I have been interested for a long time in viewing computations from an algebraic perspective. Thus a computation has to respect the operations and axioms of a specific algebraic structure, which may be Boolean algebra or a polynomial ring, for example. It can be shown that in this formulation completeness phenomena abound also, analogous to NP-completeness, and can be formulated in purely algebraic terms instead of Turing machines, in terms of which these concepts had been discovered originally. The main motivation of the algebraic approach is to bring a wider set of mathematical tools to bear on the major open problems of computational complexity. More recently, I have pursued, more specifically, the "holographic" avenue which is formulated more within linear algebra, and in which one expects questions to be mathematically more tractable.