Random Matrix Theory and central vanishing of L-functions

The American Institute of Mathematics

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Corrections and new material are welcomed and can be sent to workshops@aimath.org

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A.1 c_E, the predicted constant in the frequency of rank two in the family of quadratic twists of E

For a fixed elliptic curve E we conjecture that there are asymptotically

$$c_E X^{3/4} (log X)^{-5/8}$$

prime discriminants $p \leq X$ for which the twisted elliptic curve E_p has rank 2. Can we conjecture a formula for c_E? There are examples for which $c_E = 0$. Can we give a criterion for when that happens?

A.2 Frequency of rank 3 in a family of quadratic twists of a fixed elliptic curve

For how many fundamental discriminants d with $|d| \leq X$ does the quadratically twisted elliptic curve E_d have rank 3? This question seems to have resisted modeling by random matrix theory. Conjectures have ranged all the way from $x^{1/4}$ all the way up to $x^{1-\epsilon}$. The data sets of Watkins and of Delaunay and the examples of Rubin and Silverberg may give some hints.

A.3 Quadratic twists for higher weight forms

Random matric theory seems to suggest that quadratic twists of L-functions for weight 4 forms will have double vanishing at the critical point for about $x^{1/4}$ discriminants $d \leq x$, and only finitely many such for weight 6 and above. There is at least one set of data available to test this conjecture.

A.4 Twists of elliptic curve L-functions by cubic and higher order characters

David, Fearnley, and Kisilevsky have made conjectures about the frequency of vanishing of cubic and higher twists of a fixed elliptic curve. For cubic, the conjecture is that about $D_E X^{1/2} \log^A X$ cubic twists with conductor up to X will vanish for a certain A.

- Can we conjecture the constant D_E here?
- Is there any variation for moduli in arithmetic progressions (as there is in the quadratic case)?
- Is there any sensible way to interpret the collection of special values of cubic twists of a fixed L-function (as in the quadratic case where coefficients of half-integral weight forms give the special values)?

See the website[30] by Fearnley and Kisilevsky for more information and lots of data on these questions.

A.5 Extremely large ranks

Ulmer and others have shown that over function fields there is a sequence of integers N and elliptic curves E_N of conductor N so that the rank of E_N is greater than or equal to

$$c \frac{\log N}{\log \log N}$$

[30] page 9, *Second order vanishing of higher order twists (Fearnley and Kisilevsky)*

Does the same hold for number fields? Can random matrix theory shed any light on this question?

A.6 Real quadratic twists

Mao and and Baruch and also Gonzalo have found formulas to compute the critical values of twisted L-functions $L_E(s, \chi_d)$ for real quadratic characters. CAn one find the values of c_E here?

A.7 Siegel Modular forms

Consider the spinor zeta-function of a degree 2 weight 2 Siegel modular form. This would be a degree 4 L-function in the Selberg Class terminology. Bocherer has conjectured a special value for the central value of its quadratic twist by a fundamental discriminant d. This conjecture implies a discretization of these central values, perhaps by $1/|d|$. Can random matrix theory be used to predict the frequency of vanishing to order two of the L-functions of these quadratic twists? What are the arithmetic consequences of such vanishing?

It would be great to produce a concrete example and test it. What is the first level for which a weight 2 Siegel modular form (of degree 2) exists? How do we find the coefficients of its spinor zeta-function. Can we compute the cenral values of the quadratic twists, either directly or by Bocherer's conjecture?

A.8 Zero statistics near the central point

In order to make predictions about central values of L-functions at the central point using random matrix theory, it is necessary to have a model for the statistical behaviour of the zeros of the L-function near that point. In various families of elliptic curve L-functions S. J. Millerhttp://arxiv.org/abs/math/0508150 has observed for low conductor repulsion of the zeros from the central point. In the limit of large conductor this repulsion disappears and the zeros should follow the distribution of eigenvalues from random matrices in $SO(2N)$. It is a topic of further investigation to understand the provenance of the finite-conductor behaviour and to model it effectively.

This behaviour is seen in families of predominantly rank zero curves, as well as in families of higher rank and a sound model may lead to insight into the frequency of curves of various rank occuring within families of elliptic curves using methods similar to those described in Frequency of Vanishing[24].

Chapter B: Background information

B.1 Quadratic twists

Given a newform $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ of weight k and level N we can form its L-function

$$L_f(s) = \sum_{n=1}^{\infty} \frac{a_n/n^{(k-1)/2}}{n^s}.$$

This L-function satisfies a functional equation under $s \to 1-s$ so that the $s = 1/2$ is the center of the critical strip.

[24] page 17, *Frequency of order two vanishing of quadratic twists (Conrey, Keating, Rubinstein, and Snaith)*

Let d be a fundamental discriminant and let $\chi_d(.)$ be the real quadratic character associated with d. Thus, χ_d is the Kronecker symbol $\chi_d(n) = \left(\frac{d}{n}\right)$. Alternatively, χ_d is a real, primitive character with modulus $|d|$; this completely specifies χ_d except when $8 \mid d$, in which case there are either 0 or 2 such characters. If we note that χ_d is an even character when $d > 0$ and is odd when $d < 0$ then the ambiguity is removed. Finally, we note that the Dedekind zeta-function associated with the field $K = Q(\sqrt{d})$ is given by

$$\zeta_K(s) = \zeta(s)L(s, \chi_d).$$

We are interested in the order of vanishing of the quadratic twist

$$L_f(s, \chi_d) = \sum_{n=1}^{\infty} \frac{a_n \chi_d(n)}{n^s}$$

at the center $s = 1/2$.

If f is a modular form of weight 2 associated with an elliptic curve E/Q given by $E : y^2 = x^3 + Ax + B$ then the L-function of E is $L_E(s) = L_f(s)$ and the twisted L-function $L_E(s, \chi_d)$ is just the L-function of another elliptic curve, called the twisted elliptic curve $E_d : dy^2 = x^3 + Ax + B$. Then our question about the order of the zero of $L_E(s, \chi_d)$ is, under the asumption of the Birch and Swinnerton-Dyer conjecture, the same as the question of the distribution of ranks in the family E_d of twisted elliptic curves

B.2 Imaginary quadratic twists

Let $\chi_d(n)$ with $d < 0$ be character associated with the imaginary quadratic field $Q(\sqrt{d})$. Let f be a newform of even weight k and level N which has integral coefficients. Kohnen and Zagier showed, after work of Waldspurger, that

$$L_f(1/2, \chi_d) = \kappa_f \frac{c_{|d|}^2}{d^{(k-1)/2}}$$

where $\kappa_f > 0$ depends only on f and where $c_{|d|}$ is an integer.

In fact,

$$g(z) = \sum_{d<0} c_{|d|} e(|d|z)$$

is a cusp form of (half-integral) weight $(k+1)/2$ and level $4N$.

B.2.a Weight 2, Level 11. The unique newform of weight 2 and level 11 is given by

$$f(z) = \eta(z)^2\eta(11z)^2 = q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{11n})^2$$

where $q = e(z) = e^{2\pi i z}$. Thus

$$\begin{aligned} f(z) &= q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 - 2q^9 - 2q^{10} \\ &\quad +q^{11} - 2q^{12} + 4q^{13} + 4q^{14} - q^{15} - 4q^{16} \\ &= \sum_{n=1}^{\infty} a_n q^n. \end{aligned}$$

This modular form is associated with the elliptic curve $E_{11} : y^2 + y = x^3 - x^2$. The L-function associated with f is

$$L_{11}(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{s+1/2}}.$$

For $d < 0$, we have, by the formula of Kohnen and Zagier, as computed by Rodriguez-Villegas using the algorithm of Gross,

$$L_{11}(/12, \chi_d) = \kappa_{11} \frac{c_{11}(|d|)^2}{|d|^{1/2}}$$

where $\kappa_{11} = 2.917633\ldots$ and where

$$g(z) = \sum_{n=1}^{\infty} c_{11}(n) q^n = (\theta_1(q) - \theta_2(q))/2$$

with

$$\theta_1(q) = \sum_{\substack{(x,y,z)\in Z^3 \\ x\equiv y \mod 2}} q^{x^2+11y^2+11z^2}$$

and

$$\theta_2(q) = \sum_{\substack{x\equiv y \mod 3 \\ y\equiv z \mod 2}} q^{(x^2+11y^2+33z^2)/3}$$

B.3 Brumer's upper bound for the rank

Theorem. Let $L_E(s)$ be the L-function associated with an elliptic curve of conductor N. Let r be the order of vanishing at the center of the critical strip. If RH holds for L_E, then

$$r \leq (1/2 + o(1)) \frac{\log N}{\log \log N}$$

as $N \to \infty$.

Proof. Let $F(x) = 1 - |x|$ be supported on $[-1, 1]$. The Fourier transform of this is $G(t) = (2/t^2)(1 - \cos t)$, which is everywhere nonnegative. By the explicit formula for this L-function (see page 219 of Mestre), we have

$$\sum_{\rho} \Phi_\lambda(\rho) + 2 \sum_{p^m} b(p^m) F_\lambda(m \log p) \frac{\log p}{p^m} = \log N - 2 \log 2\pi - 2 \int_0^{\infty} \frac{F_\lambda(x)}{e^x - 1} - \frac{e^{-x}}{x}\, dx,$$

where $b(p^m)$ is the Frobenius trace (i.e. minus the coefficient of the logarithmic derivative of $L_E(s)$. The coefficient is not normalized and so, by Hasse or Weil, $b(m)$ has size around $\sqrt{m}$ for m a prime power.), $\Phi(s) = G(-i(s - 1/2))$ and $F_\lambda(x) = F(x/\lambda)$ so that $G_\lambda(t) = \lambda G(\lambda t)$ where λ is a parameter. The integral tends to γ as $\lambda \to \infty$. The positivity of Φ (under GRH) and the Weil bound for $b(p^m)$ give us that $\lambda r G(0) \leq \log N + 4 \sum_{p^m \leq e^\lambda} \frac{\log p}{p^{m/2}} + O(1)$; the sum is easily bounded by $2e^{\lambda/2} \log 3$. By taking $\lambda = 2 \log \log N - \log \log \log N$, and noting that $G(0) = 1$, we thus get that $r \leq (1/2 + o(1)) \frac{\log N}{\log \log N}$.

Chapter C: Links to data

The items in this section include lots of experimental data relevant to the focus of this workshop.

C.1 Tornaria/Rubinstein data for vanishing, thousands of curves

Data[1] for the twists with $|d| < 10^8$ for thousands of elliptic curves based on the Tornaria-Villegas ternary forms.

Notes[2]

C.2 Second order vanishing for real quadratic twists of the weight 4 level 7 Hecke cuspform

Data[3] for quadratic twists of the weight 4 level 7 Hecke cuspform f.

This form can be given as $f = q - q^2 - 2q^3 - 7q^4 + 16q^5 + 2q^6 - 7q7 + \dots$.

Here are the first few normalized values of the twisted L-series. This should be cmpared with Table 3.2 of the Rosson Tornaria paper which starts on page 315 of the Newton Proceedings. Note, that in the data below the discriminants divisible by 7 have been omitted, and also the entries have been scaled back by a factor of 7. And the entry corresdponding to $d = 1$ is also not present. Below are the beginnings of the data. (The actual data set has millions of entries.)

```
8 1
29 -4
37 0
44 4
53 4
57 -2
60 -6
65 2
85 -4
88 6
92 0
93 -4
109 4
113 2
120 -2
137 -4
141 12
149 0
156 14
165 -12
172 -4
177 6
184 -12
```

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[1] http://pmmac03.math.uwaterloo.ca/∼mrubinst/L$_f unction_p ublic/VALUES/DEGREE_2/ELLIPTIC/QUADRATIC_$
[2] http://pmmac03.math.uwaterloo.ca/∼mrubinst/L$_f unction_p ublic/VALUES/DEGREE_2/ELLIPTIC/QUADRATIC_$
[3] http://pmmac03.math.uwaterloo.ca/∼mrubinst/L$_f unction_p ublic/VALUES/DEGREE_2/WEIGHT_{4e} xample$

```
193 6
197 8
```

C.3 Rank one quadratic twists, Delaunay and Roblot

We give some numerical data on the quadratic twists by negative discriminants of the first smallest (for the conductor) elliptic curves. For each curve (11a1, 14a1, 15a1 and 17a1), the file contains a PARI/GP vector W of length 1500000 such that W[b] is non-zero iff -b is a fundamental discriminant and satisfying additional conditions so that the sign of the functional equation is -1. In that case, W[b] then contains a vector whose entries are: 1) the short ellinit of the minimal Weierstrass model of the quadratic twist, 2) the product of the Tamagawa numbers, 3) a generator of the Mordell-Weil group, 4) the analytic order of the Tate-Shafarevich group, 5) the index of the subgroup generated by torsion and the Heegner point used in the computation. Note that if the analytic rank of the quadratic twist is greater than 1 then the last three entries are zero.

Data[4]

See the bottom of the page.

C.4 Watkin's lists of twists with odd ranks at least 3

This data[5] contains lists of which quadratic twists of given curves (up to 100A) have odd parity, satisfy a Heegner hypothesis, and have L'(E_d, 1) = 0.

For example, here is the beginning of his list of d for which the quadratic twist of the L-function associated with the elliptic curve of conductor 11 vanishes to odd order at least 3:

824	1007	1799	4399	8483	11567	14791	15487
15659	15839	16463	17023	17927	18543	19807	20247
20895	20984	23503	26039	30263	32551	32808	33887
34143	34663	36959	38807	39903	39947	40103	41335
41663	42919	43903	44527	45407	45524	45671	47759
50783	54247	55919	59287	60968	64103	64415	65236
67031	67063	67128	68855	70143	75688	76952	77987
78548	81028	81287	82631	83767	84292	84567	84983
87391	87463	90943	91640	91768	91955	92263	96168
97352	98116	99743	105047	107543	107912	108008	108539
114347	114872	120463	120859	121448	122447	122623	125879
129983	130008	130767	131423	131911	132923	134063	134312
135167	135176	135847	136199	138067	140223	140696	141448
145439	145559	146863	148103	150347	150367	150647	151879
152215	154343	154824	155359	157055	160388	160519	162799
164903	165711	166751	176183	176239	177935	178679	179467
181048	186711	190043	190119	192935	195263	195368	198527
198824	202663	203656	204503	204731	205784	207935	209919

[4]http://math.univ-lyon1.fr/~roblot/tables.html

[5]http://www.maths.bris.ac.uk/~mamjw/ALL.tar

C.5 Vanishing for twists by $d \equiv 0$ modulo p

Rubinstein's web site[6] has data which looks at the ratio of vanishing with $d = 0$ mod p compared to all d and shows a connection to the group structure of $E(F_p)$. The data is sorted in increasing value of p * (ratio of vanishing). Smaller values tend to go with cyclic groups, larger values with products of two cyclic groups.

The number in the filenames indicates the total number of twists considered in the data set.

For example, here is the header in the E_{11} file

column 1: p * (vanishing = 0 mod p)/total

column 2: p

column 3: a_p

column 4: $N_p = p + 1 - a_p$

column 5: group structure of $E(F_p)$

and here are some initial entries:

0.6576839438 3 -1 5 : cyclic
0.6594793716 1193 -21 1215 : cyclic
0.6755940597 1301 27 1275 : 255 x 5
0.6793569962 1549 -15 1565 : cyclic
0.7005016233 1637 33 1605 : cyclic
0.7059290235 1409 -15 1425 : cyclic

In the last line here, the 1409 indicates that we are twisting E_{11} by discriminants divisible by 1409. We found that the proportion of these that had rank even positive rank is about 0.7059/1409. The value of a_{1409} for $E_1 1$ is -15. The value of N_{1409} is 1425. The group of points of E_{11} over the finite field of size 1409 is cyclic.

C.6 Second order vanishing of higher order twists (Fearnley and Kisilevsky)

A very nice web site by Jack Fearnley and Hershey Kisilevsky with data on vanishing of central values of cubic and higher order twists of L-functions associated with elliptic curves. Web page[7]

C.7 Cubic twist of the L-function of E over a quadratic extension (Watkins)

Start with E/Q, base extend to an imaginary quadratic field of class number 3, and then twist by cubic characters. How often does the L-function vanish? Donnelly and Watkins investigate this and here[8] is a description of their data.

C.8 Coefficients for the first few spinor zeta-functions (by Kuss)

Coefficients[9] for the first few spinor zeta-functions and cental values of their twists.

[6] http://www.math.uwaterloo.ca/∼mrubinst/vanishing$_{a}nd_{g}p$

[7] http://www.dms.umontreal.ca/∼jack/local/

[8] http://www.maths.bris.ac.uk/∼mamjw/blurb.pdf

[9] http://www.aimath.org/conrey/banff/Kuss$_{d}ata.tar.gz$

For example here are the first few coefficients for the weight 20 degree 2 spinor zeta-fucntion:

```
#
#  Tabelle der Eigenwerte von
#
# -1*2^9*3^2*5*7*11*V(phi10) * V(phi10)+1/2*V(phi12*E8)+1/2*V(phi10*E10)
#
#  modulo 769511056930104637822105478184684011205425086641080997289138168656358804425712
#
#

lambda(2) = -840.960
lambda(3) =  346.935.960
lambda(5) = -5.232.247.240.500
lambda(7) =  2.617.414.076.964.400
lambda(11) =  1.427.823.701.421.564.744
lambda(13) = -83.773.835.478.688.698.980
lambda(17) =  14.156.088.476.175.218.899.620
lambda(19) =  146.957.560.176.221.097.673.720
lambda(23) = -7.159.245.922.546.757.692.913.520
lambda(29) =  1.055.528.218.470.800.414.110.149.180
lambda(31) =  4.031.470.549.468.367.403.585.068.224
lambda(37) = -154.882.657.977.740.251.483.442.365.940
lambda(41) =  1.126.683.124.934.949.617.518.831.346.964
lambda(43) =  74.572.686.686.194.644.813.168.430.600
lambda(47) = -13.773.335.595.379.978.013.820.602.730.720
lambda(53) =  29.292.488.702.536.161.643.591.933.657.260
lambda(59) =  521.943.213.201.995.351.655.113.144.025.960
lambda(61) =  896.978.197.899.858.751.399.574.623.768.444
lambda(67) = -2.921.787.486.641.381.474.027.809.454.434.280
lambda(71) = -10.040.723.756.183.052.209.438.026.581.108.816
lambda(73) =  35.758.360.743.202.050.929.441.049.170.653.780
lambda(79) = -162.246.739.774.459.334.098.015.102.062.271.520
lambda(83) = -206.892.964.328.868.666.394.961.575.845.492.360
lambda(89) =  680.395.062.782.220.767.996.769.753.143.492.340
lambda(97) =  2.332.265.955.650.189.235.208.550.925.056.957.380
lambda(101) =  4.386.172.918.053.625.462.559.466.358.055.947.404
lambda(103) =  32.195.086.396.948.327.734.208.192.573.193.192.560
lambda(107) = -24.228.077.499.906.736.103.239.295.291.195.489.400
lambda(109) = -12.955.699.549.333.026.964.322.913.228.574.296.740
lambda(113) =  36.255.949.371.446.088.080.680.636.143.504.043.620
lambda(127) = -361.342.894.775.546.586.251.665.886.043.645.512.960
lambda(131) = -2.053.619.322.749.000.749.691.039.889.490.669.814.376
lambda(137) = -3.744.460.506.310.723.939.277.414.819.956.659.325.740
lambda(139) =  419.297.609.100.113.389.140.294.392.040.631.725.640
lambda(149) =  10.769.495.917.074.628.858.977.625.085.917.067.553.900
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```
lambda(151) =  9.334.247.630.257.185.873.211.774.091.931.905.801.104
lambda(157) =  33.793.021.179.199.016.867.594.391.971.251.561.680.700
lambda(163) = -110.850.825.156.572.723.151.988.432.773.370.438.791.080
lambda(167) = -118.679.333.472.681.624.640.702.222.790.918.993.560.080
lambda(173) =  46.749.322.156.688.096.269.201.140.297.725.725.372.380
lambda(179) =  104.233.219.767.057.822.550.544.911.884.186.972.529.080
lambda(181) =  207.762.909.808.835.396.767.334.770.633.028.101.969.324
lambda(191) = -63.517.483.904.754.733.670.967.681.976.125.665.751.936
lambda(193) = -267.251.286.131.268.420.110.302.359.162.667.934.949.500
lambda(197) = -255.034.267.090.827.972.833.444.866.782.055.909.482.420
lambda(199) =  3.624.095.258.996.012.770.479.419.815.087.685.961.665.200
lambda(211) =  9.782.628.970.544.607.179.115.817.803.912.815.906.919.544
lambda(223) =  7.633.350.373.112.207.921.215.176.727.261.847.796.623.680
lambda(227) =  3.690.457.413.425.483.396.410.498.083.461.810.010.273.240
lambda(229) = -9.459.107.646.730.616.297.026.260.721.656.143.628.885.620
lambda(233) = -6.518.233.529.510.248.143.084.719.131.742.966.396.228.460
lambda(239) = -174.289.033.715.822.749.051.787.796.341.723.196.742.769.760
lambda(241) =  165.411.807.125.763.090.448.015.609.164.763.861.692.881.764
lambda(251) =  10.040.525.828.214.215.237.077.359.707.488.429.149.078.504
lambda(257) =  559.093.438.914.779.356.899.053.428.598.510.513.591.568.900
lambda(263) = -1.093.446.135.900.068.391.671.153.374.335.286.886.635.158.480
lambda(269) = -779.319.800.265.343.124.419.069.257.884.629.979.797.575.780
lambda(271) =  2.126.139.242.814.193.873.497.724.221.863.293.610.241.445.984
lambda(277) = -1.829.398.521.407.900.555.919.736.166.452.464.004.650.368.660
lambda(281) =  873.606.030.193.110.809.794.772.171.610.387.367.679.876.724
lambda(283) =  2.284.669.073.643.510.737.359.980.059.014.097.257.668.316.840
lambda(293) = -1.002.663.569.785.176.779.113.280.997.156.603.069.381.798.900
lambda(307) = -13.034.670.684.793.860.775.645.487.512.403.484.297.745.757.000
lambda(311) = -7.787.112.252.426.689.018.271.832.043.149.843.510.894.843.056
lambda(313) = -11.591.518.908.086.637.517.572.515.263.901.895.018.178.223.180
lambda(317) = -9.179.771.636.920.468.169.855.409.559.744.825.026.721.371.780
lambda(331) = -213.447.033.740.089.463.462.053.489.746.773.284.457.099.576
lambda(337) =  84.669.849.416.835.714.006.952.735.473.040.220.442.656.890.660
lambda(347) =  230.458.399.280.174.855.021.546.789.850.838.785.135.022.883.880
lambda(349) = -185.619.024.402.281.287.631.814.735.712.109.460.539.452.952.900
lambda(353) =  2.598.771.740.172.754.824.454.272.724.943.092.027.018.821.060
lambda(359) =  117.608.353.485.511.419.149.405.458.741.378.983.255.079.643.760
lambda(367) = -367.777.575.361.137.277.090.843.735.111.125.772.482.655.971.680
lambda(373) = -322.660.653.230.457.762.543.907.839.235.182.295.913.572.766.420
lambda(379) = -79.805.169.831.994.879.651.985.700.137.963.052.252.556.349.720
lambda(383) =  12.745.650.135.391.529.798.274.593.543.759.038.771.196.893.440
lambda(389) =  1.293.552.421.949.295.265.163.351.505.836.527.141.014.458.632.140
lambda(397) = -1.679.687.580.771.102.854.867.693.108.890.062.219.800.095.794.020
lambda(401) =  2.858.201.743.929.466.986.375.501.968.188.691.120.029.233.629.604
lambda(409) =  348.729.101.349.455.034.615.416.081.981.576.296.433.036.467.060
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```
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lambda(431) =  5.036.121.842.030.593.464.215.094.552.734.870.331.960.809.377.824
lambda(433) =  7.234.292.070.823.611.620.133.890.216.498.300.909.312.543.744.740
lambda(439) =  738.529.111.605.902.162.612.444.304.773.541.252.611.537.231.440
lambda(443) = -2.257.815.213.371.794.341.783.826.716.082.311.901.264.623.279.000
lambda(449) =  13.156.299.820.810.809.466.350.508.583.994.241.855.754.719.373.700
lambda(457) =  6.670.680.987.546.252.887.830.377.479.657.164.176.483.836.713.300
lambda(461) = -8.923.632.858.941.625.315.095.184.613.338.776.729.316.895.561.956
lambda(463) = -696.067.706.851.539.160.752.742.331.382.008.462.070.343.189.280
lambda(467) =  48.783.493.386.294.650.412.247.653.778.029.736.614.224.822.438.520
lambda(479) = -52.376.572.802.707.441.817.786.512.485.427.720.744.345.040.253.120
lambda(487) =  884.502.023.850.550.023.970.426.880.973.342.613.132.327.918.960
lambda(491) = -37.978.790.125.817.668.114.350.835.447.222.448.242.534.457.039.736
lambda(499) = -4.203.062.739.409.816.448.391.998.976.406.504.126.239.608.333.000
lambda(503) = -74.213.760.614.913.224.368.709.462.739.792.914.274.629.395.925.040
lambda(509) = -94.930.208.442.699.841.613.845.682.445.466.671.917.610.518.894.340
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lambda(523) =  313.930.583.083.149.945.729.047.210.822.295.717.713.472.089.329.480
lambda(541) = -846.417.121.476.315.261.672.039.135.945.854.951.813.046.352.196.036
lambda(547) =  179.035.567.841.599.802.665.704.541.597.417.609.048.767.240.360.280
lambda(557) =  336.270.132.565.793.623.424.075.329.900.366.532.220.446.392.225.500
lambda(563) = -53.231.544.021.380.199.976.397.997.303.703.098.799.621.602.532.680
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lambda(571) =  5.695.029.846.320.377.697.851.385.204.354.500.168.481.761.038.184
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lambda(599) = -305.520.766.836.151.676.559.795.767.188.491.294.933.493.724.156.400
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lambda(613) =  365.069.572.451.456.821.029.003.100.393.286.737.339.054.568.468.620
lambda(617) =  4.744.987.063.843.265.040.759.946.301.141.998.615.248.108.098.300.820
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lambda(683) = -30.161.466.767.718.939.050.406.186.475.363.432.102.470.557.933.312.760
lambda(691) = -38.999.485.389.650.722.748.821.201.910.568.567.015.164.335.393.124.936
lambda(701) = -40.624.891.939.967.321.528.907.667.172.011.018.469.627.869.398.968.196
lambda(709) = -56.856.096.442.287.109.119.788.176.503.051.232.034.090.388.493.065.140
```

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lambda(739) = -101.944.634.031.348.580.562.020.592.193.446.815.898.762.689.737.694.760
lambda(743) = -10.450.835.913.190.722.221.709.167.966.608.064.844.152.967.358.381.200
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lambda(769) =  83.866.146.020.804.439.049.293.616.618.297.545.603.824.725.747.017.220
lambda(773) =  84.471.092.544.936.412.635.213.833.928.284.723.948.336.583.838.987.980
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lambda(839) =  794.373.884.628.008.486.292.192.242.928.704.313.260.247.475.384.977.840
lambda(853) =  360.886.834.243.802.181.174.849.708.947.408.775.243.507.497.880.238.060
lambda(857) = -147.310.889.855.119.834.574.086.892.045.458.202.371.953.953.179.269.900
lambda(859) =  89.368.832.236.809.785.565.865.416.378.424.748.312.340.025.277.214.760
lambda(863) = -2.333.780.259.369.874.453.904.462.038.337.504.435.007.101.291.746.514.880
lambda(877) =  45.365.619.315.356.974.884.365.276.786.671.362.025.604.628.208.456.540
lambda(881) =  6.921.921.598.802.920.445.308.726.020.098.848.144.679.154.512.431.361.124
lambda(883) =  6.314.712.712.082.575.161.074.583.362.460.903.180.280.713.694.684.880.440
lambda(887) = -2.062.615.119.839.681.438.263.706.452.588.931.073.473.471.229.237.096.240
lambda(907) = -2.297.300.139.775.143.823.836.589.429.818.796.449.519.931.962.494.799.800
lambda(911) =  2.502.332.172.299.861.337.458.680.117.453.813.199.170.053.553.847.701.344
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lambda(941) =  6.910.242.774.849.130.535.729.742.154.316.194.098.929.850.966.529.113.564
lambda(947) = -3.922.053.237.528.471.526.894.368.916.080.795.990.942.198.787.169.614.920
lambda(953) = -9.744.586.276.127.704.137.071.387.409.413.743.671.485.927.494.322.795.340
lambda(967) =  7.702.986.144.431.860.997.026.595.809.197.973.943.227.493.084.584.377.520
lambda(971) =  25.989.700.002.773.199.653.825.944.249.995.706.984.782.097.406.732.387.78
lambda(977) =  38.480.875.858.561.905.041.313.051.867.595.220.636.913.680.079.843.016.74
lambda(983) = -32.423.811.165.177.479.059.590.178.700.572.318.345.232.334.028.001.310.96
lambda(991) = -36.093.870.763.467.971.560.763.644.574.811.761.504.935.884.691.226.092.73
lambda(997) = -11.827.252.416.410.698.465.141.694.507.545.048.712.780.663.614.521.544.82
#
#
lambda(2^2) =  248.256.200.704
lambda(3^2) = -452.051.040.393.665.991
lambda(5^2) = -94.655.785.156.653.029.446.859.375
```

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lambda(7^2) = -5.501.629.950.184.780.949.434.983.315.951
lambda(11^2) = -126.258.221.861.417.704.499.584.077.355.164.268.151
lambda(13^2) =  2.528.254.555.352.510.520.887.488.261.241.887.242.369
lambda(17^2) =  262.144.933.510.286.336.089.464.293.262.250.165.947.750.889
lambda(19^2) = -283.417.759.450.334.375.466.210.009.895.464.677.379.295.086.759
lambda(23^2) =  127.862.428.522.278.879.932.688.110.084.314.434.400.497.569.566.129
lambda(29^2) =  408.550.299.154.535.330.723.926.336.201.059.419.422.405.306.949.883.361
lambda(31^2) = -9.417.686.481.892.622.568.784.061.821.415.683.057.728.289.096.885.473.47
lambda(37^2) =  4.270.657.975.661.931.417.960.508.434.757.260.969.748.219.593.839.247.06
lambda(41^2) =  129.620.395.091.878.626.890.240.343.719.327.738.119.688.391.311.944.613.
lambda(43^2) = -2.118.391.905.744.174.698.890.014.439.813.915.105.652.042.393.393.982.40
lambda(47^2) =  10.717.867.956.150.312.430.187.083.192.735.560.357.439.349.298.395.760.6
lambda(53^2) = -6.359.983.052.359.692.969.866.068.986.893.310.598.482.880.773.029.488.94
lambda(59^2) =  159.291.906.542.794.821.742.879.348.124.552.646.753.906.149.121.778.952.
lambda(61^2) = -653.805.853.261.332.407.170.328.486.766.159.640.869.797.840.457.778.124.
lambda(67^2) =  25.254.882.862.606.589.034.647.035.623.760.404.781.292.970.925.413.106.2
lambda(71^2) =  173.803.021.217.204.348.159.401.727.499.142.397.151.285.125.090.266.111.
lambda(73^2) =  244.586.510.121.809.529.678.274.464.585.042.290.025.095.172.202.619.886.
lambda(79^2) = -4.991.297.766.303.348.937.077.560.233.832.507.863.616.069.468.135.037.03
```

C.9 Computing the analytic rank (program from S.J. Miller)

The auxiliary file attached to this article is a package for PARI written by S. J. Miller to compute the analytic rank of a curve. The instructions on how to use it at the command line in PARI follow:

to load in the program arank.gp for command line usage:

årank.gp

now in PARI let E be an elliptic curve:

E = ellinit([0,0,1,-7,6])

to find the rank simple type at the command prompt

ellarank(E)

here are some curves with given ranks:

rank 3: [0,0,1,-7,6]

rank 4: [1,-1,0,-79,289]

rank 5: [0,0,1,-79,342]

rank 6: [0,0,1,-7077,235516]

If the conductor is large, the program takes a bit longer to run. For example, consider the following

E5 = ellinit([0,0,0,-15823,767122])

ellarank(E5)

It will see the rank is 5, but it will take a noticeable amount of time. 5 points on curve are (81,130) (83,160) (74,38) (71,40) (69,62) The height matrix of these points is approximately 32.5.

Chapter D: Notes from the Banff workshop

Where speakers have submitted notes to their talks, these notes can be found here[10].

Some notes and slides for various talks are also below, as well as links to relevant papers. Click here[11] for the program and abstracts to all talks.

- Henri Darmon: Shintani lifts, p-adic families and derivatives of quadratic twists: notes from the talk[12].
- Christophe Delaunay: Thoughts on c_E: notes from the talk[13].
- Eduardo Dueez, Duc Khiem Huynh, Jon Keating and Steven J. Miller: Finite conductor models for zeros of elliptic curves: slides[14].
  Relevant papers:
  Investigations of zeros near the central point of elliptic curve L-functions[15] (Miller and Dueez)
  Applications of the L-functions ratios conjecture[16](Conrey and Snaith)
- Zhengyu Mao: Shimura correspondence and computation of L-values: notes from the talk[17].
- Mark Watkins: Extemporaneous talk on random matrix theory and symmetric powers: notes from the talk[18].

There was also a discussion session session on open, important problems. Notes are available here[19]. Additional notes from the workshop can also be found here[20].

Chapter E: Proceedings, papers and reviews

- Ehud Moshe Baruch, Zhengyu Mao. Central value of automorphic $L-$functions. [arxiv:math.NT/0301115]
- Brian Conrey, Jon Keating, Mike Rubinstein, and Nina Snaith. On the frequency of vanishing of quadratic twists of modular L-functions. [arxiv:math.NT/0012043]
- Chantal David, Jack Fearnley, and Hershey Kivilevsky. On the vanishing of twisted L-functions of elliptic curves. David's Home Page[21].
- Christophe Delaunay. Heuristics on Tate-Shafarevitch Groups of Elliptic Curves Defined over Q. [2003a:11065]
- Noam Elkies and Mark Watkins. Elliptic curves of large rank and small conductor.[22]
- Fernando Gouvea and Barry Mazur. The square free sieve and the rank of elliptic curves [92b:11039] J. Amer. Math. Soc. 4 (1991), no. 1, 1–23.

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[10]http://www.maths.bris.ac.uk/∼mancs/banff.html
[11]http://www.aimath.org/∼aimath/WWN/rmtapplications/Programme07w5114.pdf
[12]http://www.aimath.org/∼aimath/WWN/rmtapplications/Darmon.pdf
[13]http://www.aimath.org/∼aimath/WWN/rmtapplications/Delaunay.pdf
[14]http://www.aimath.org/∼aimath/WWN/rmtapplications/Banff$_D HKM.pdf$
[15]http://arxiv.org/abs/math/0508150
[16]http://arxiv.org/abs/math/0509480
[17]http://www.aimath.org/∼aimath/WWN/rmtapplications/Mao.pdf
[18]http://www.aimath.org/∼aimath/WWN/rmtapplications/Watkins.pdf
[19]http://www.aimath.org/∼aimath/WWN/rmtapplications/Discussion.pdf
[20]http://www.maths.bris.ac.uk/∼mancs/banff.html
[21]http://cicma.mathstat.concordia.ca/faculty/chantal/publi.html
[22]http://www.math.psu.edu/watkins/antsVI.ps

- Roger Heath-Brown. The Size of Selmer Groups for the Congruent Number Problem II [95h:11064] Invent. Math. 118 (1994), no. 2, 331–370.
- Karl Rubin and Alice Silverberg. Rank frequencies for quadratic twists of elliptic curves. [arxiv:math.NT/0010056]
- Ranks of elliptic curves, by Karl Rubin; Alice Silverberg. article[23]
- Doug Ulmer. Elliptic curves with large rank over function fields. [arxiv:math.NT/0109163]
- Mark Watkins. Rank distribution in a family of cubic twists.[24]
- D. Zagier and G. Kramarz. [90d:11072] Numerical investigations related to the L-series of certain elliptic curves. J. Indian Math. Soc. (N.S.) 52 (1987), 51–69 (1988).
- Average ranks of elliptic curves: Tension between data and conjecture by Baur Bektemirov; Barry Mazur; William Stein; Mark Watkins. Bulletin article[25]

E.1 Proceedings of the Newton Institute workshop on RMT and ranks of elliptic curves

Download proceedings[26] in pdf format of the Clay workshop at the Newton institute, February 2004, edited by Conrey, Farmer, Mezzadri, and Snaith.

Contents

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[23] http://www.ams.org/journal-getitem?pii=S0273-0979-02-00952-7

[24] http://www.math.psu.edu/watkins/papers.html

[25] http://www.ams.org/bull/2007-44-02/S0273-0979-07-01138-X/home.html

[26] http://www.aimath.org/conrey/proceedings.pdf

Vanishing of L-functions of elliptic curves over number Fields 247 C. David, J. Fearnley and H. Kisilevsky

Shimura correspondence, and twists Computing central values of L-functions 260 F. Rodriguez-Villegas

Computation of central value of quadratic twists of modular Lfunctions 273 Z. Mao, F. Rodriguez-Villegas and G. Tornaria

Examples of Shimura correspondence for level p2 and real quadratic twists 289 A. Pacetti and G. Tornaria

Central values of quadratic twists for a modular form of weight 4 315 H. Rosson and G. Tornaria

Global structure: Sha and descent Heuristics on class groups and on Tate-Shafarevich groups 323 C. Delaunay

A Note on the 2-Part of X for the Congruent Number Curves 341 D.R. Heath-Brown

2-Descent Through the Ages 345 P. Swinnerton-Dyer

E.2 Frequency of order two vanishing of quadratic twists (Conrey, Keating, Rubinstein, and Snaith)

Brian Conrey, Jon Keating, Mike Rubinstein, and Nina Snaith. On the frequency of vanishing of quadratic twists of modular L-functions. [arxiv:math.NT/0012043]

This paper was the first to use random matrix theory to investigate the frequency of higher ranks in a family of elliptic curves.

E.3 Frequency of order two vanishing of cubic twists (David, Fearnley, Kisilevsky)

Chantal David, Jack Fearnley, and Hershey Kivilevsky. On the vanishing of twisted L-functions of elliptic curves. David's Home Page[27].

This paper considers the central vanishing of an elliptic curve twisted by a complex Dirichlet character of small order. One might think such L-series never vanish. This paper has very nice theoretical number theory results, numerical results, and heuristics inspired by random matrix theory.

E.4 Frequency of order 3 vanishing in the family of quadratic twists of a fixed elliptic curve (Watkins)

This[28] is a new preprint of Mark Watkins which reports on his experiments having to do with the frequency of rank 3 curves within the family of quadratic twists of a fixed elliptic curve.

E.5 Zero statistics (Miller, Duenez)

Papers on zero statistics near the central point of elliptic curve L-functions:

[1] S. J. Miller[29] Investigations of Zeros Near the Central Point of Elliptic Curve L-Functions (with an appendix by Eduardo Duenez), Experimental Mathematics, vol 15 (2006), no. 3, 257–279.

[27] http://cicma.mathstat.concordia.ca/faculty/chantal/publi.html
[28] http://www.maths.bris.ac.uk/∼mamjw/r3.pdf
[29] http://arxiv.org/abs/math/0508150

Numerical investigations of how the distribution of the first few zeros above the central point depends on the rank of the family and the size of the conductor, and a description of the independent and interaction models for zeros near the central point.

[2] S. J. Miller[30] 1- and 2-Level Densities for Rational Families of Elliptic Curves: Evidence for the Underlying Group Symmetries, Compositio Mathematica, vol 140, Issue 4, July 2004, 952–992

An analysis of the behavior of the zeros near the central point in the limit as the conductors tend to infinity.

E.6 Vanishing of quadratic twists of symmetric power L-functions (Watkins)

"Some Heuristics about Elliptic curves" by Mark Watkins. paper[31] See pages 19 - 23 for a discussion of vanishing of twists of symmetric power L-functions.

E.7 Regulators of rank 1 curves in a family of quadratic twists (Delaunay, Roblot)

A preprint[32] by Delaunay and Roblot about the regulators of rank 1 curves in a family of quadratic twists.

E.8 Siegel Modular Forms, including several expository papers, and a book

Very short[33] introduction to Siegel Modular forms by Ghitza.

Lengthier introduction[34] by van der Geer.

Numerical verification of Bocherer conjecture[35] by Kohnen and Kuss.

The book The Theory of Jacobi Forms by Eichler and Zagier (out of print) but available in scanned form here. You can either get it in three files: Part 1, 4 mg[36], Part 2, 11 mg[37], Part 3, 20 mg[38] or you can get the Whole book, 35 mg[39]

E.9 Symmetric cube L-functions as spinor zeta-functions (Schulze-Pillot)

In this note[40] Schulze-Pillot points out that Watkins symmetric cube L-functions are weight 3 spinor L-functions associated to degree 2 Siegel modular forms for an appropriate congruence subgroup.

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[30] http://arxiv.org/abs/math/0310159
[31] http://www.maths.bris.ac.uk/∼mamjw/heur.pdf
[32] http://www.aimath.org/conrey/banff/rank1$_{2}$.pdf
[33] http://www.aimath.org/conrey/banff/siegel$_{m}odular_{f}orms/ghitza$.pdf
[34] http://www.aimath.org/conrey/banff/siegel$_{m}odular_{f}orms/geer$.pdf
[35] http://www.aimath.org/conrey/banff/siegel$_{m}odular_{f}orms/kuss_{a}nd_{k}ohnen$.pdf
[36] http://www.aimath.org/conrey/banff/jacobi$_{f}orms_{p}art_{1}$.pdf
[37] http://www.aimath.org/conrey/banff/jacobi$_{f}orms_{p}art_{2}$.pdf
[38] http://www.aimath.org/conrey/banff/jacobi$_{f}orms_{p}art_{3}$.pdf
[39] http://www.aimath.org/conrey/banff/jacobi$_{f}orms_{w}hole_{b}ook$.pdf
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