The following compilation of participant contributions is only intended as a lead-in to the ARCC workshop “Representations of surface groups.” This material is not for public distribution. Corrections and new material are welcomed and can be sent to workshops@aimath.org

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Table of Contents

A. Participant Contributions ............................................. 3
   1. Baird, Tom
   2. Bonahon, Francis
   3. Charette, Virginie
   4. Choi, Suhyoung
   5. Daskalopoulos, George
   6. Garcia-Prada, Oscar
   7. Goldman, Bill
   8. Kotschick, Dieter
   9. Lawton, Sean
  10. Link, Gabriele
  11. Melnick, Karin
  12. Mondello, Gabriele
  13. Tan, Ser
  14. Toledo, Domingo
  15. Vakil, Ravi
  16. Wienhard, Anna
  17. Wilkin, Graeme
  18. Wolf, Michael
Chapter A: Participant Contributions

A.1 Baird, Tom

I have been studying the moduli space of representations, $\text{Hom}(\pi, G)/G$, when $\pi$ is the fundamental group of a nonorientable surface and $G$ is a (compact) Lie group. I am specifically interested in the cohomology ring of these spaces. My technique is to compute the $G$-equivariant cohomology of $\text{Hom}(\pi, G)$ and to use equivariant localization to obtain information about the quotient under conjugation. This has been carried out for $G = SU(2)$ but obstacles remain in the general case. One question I’ve been thinking about lately is whether the conjugation action of $G$ on $\text{Hom}(\pi, G)$ is equivariantly formal for general (compact) $G$.

In a related problem, I derived formulas for the ordinary and equivariant cohomology of (a component of) $\text{Hom}(\mathbb{Z}^n, G)$ when $G$ is compact (see math.AT/0610761), answering a question posed by Adem and Cohen (math.AT/0603197).

I am interested in this workshop as an opportunity to meet people and to broaden my perspective on the study of surface group representations. I am especially interested to learn more about the wonderful world of Higgs bundles.

A.2 Bonahon, Francis

I have had a long term interest in representations of surface groups in $SL_2(C)$, occurring in my research as the isometry group of hyperbolic 3-space. My more recent work has been focused on the quantum Teichmüller space, a non-commutative deformation of the algebra of rational functions on the representation variety of a surface group over $SL_2(C)$. I come to the workshop with two goals/expectations. The main one is to learn more about Higgs bundles, which have always been a little mysterious to me. Another one is to tap into the insights of some of the participants on conformal blocks, whose theory seems to have many points in common with my current research.

A.3 Charette, Virginie

My goal for the workshop is to gain a better understanding of the Fock-Goncharov-Penner-Thurston parametrisation of Teichmüller space and explore new applications. I am particularly interested in discussing a recent contribution of Labourie-McShane, extending the McShane-Mirzakhani identities to Hitchin representations. I am curious to see how one could transfer these ideas to settings such as affine Lorentzian or conformal structures.

A.4 Choi, Suhyoung

Currently, I am interested in finding out how much of the representation spaces can be covered by geometric configurations coming from pair of pants decompositions of the surfaces. The basic ingredients are analogous to four adjacent triangles as Goldman constructed in “Convex real projective structures on compact surfaces” J. Differential Geom. 31 (1990), no. 3, 791-845.

Using this idea, I can find cell decompositions of the characteristic variety of the $S0(3)$-representations of the fundamental group of a closed genus 2 surface. The characteristic variety of the $SU(2)$-representations branch covers this space. The method generalizes to higher-genus surfaces.
A question that I had some time ago was given a flat bundle with nonzero torsion over a surface whether it is possible to compute the Euler class from the torsion. The Lie group could be O(2) or SL(2,R). Conversely, given a flat bundle and topological informations about the bundle, can we say something about the torsion?

A.5 Daskalopoulos, George

I will contribute in two different directions:
1. Harmonic maps into homogeneous manifolds.
2. The topology of the space of SL(2,C) representations.

Of course these concepts are linked via Higgs bundles, which is one of the main concepts in this workshop.

A.6 Garcia-Prada, Oscar

I am interested in the geometry and topology of moduli spaces of representations of surface groups in Lie groups. My approach to the problem is via the theory of Higgs bundles on a Riemann surface and the use of Morse theory on the moduli space of these objects. I am currently interested in the case of Lie groups that are the isometry group of a non-compact Hermitian symmetric space.

A.7 Goldman, Bill

I am interested in the geometry, topology and dynamics of spaces of surface group representations. My interest in this subject derived from their relationship to the classification (deformation theory) of locally homogeneous (in the sense of Ehresmann and Thurston) geometric structures on surfaces. Currently I am interested in the dynamical properties of the action of the mapping class group, and the closely related question of how more refined analytic structures (such as hyper-Kahler metrics on Higgs bundle moduli spaces) depend on the conformal structure of the Riemann surface.

A.8 Kotschick, Dieter

I am interested in representations of surface groups that factor through the mapping class group of another surface. This is important in the study of surface bundles over surfaces, and of more general Lefschetz fibrations, and is related to calculations of Gromov-Thurston norms on various groups.

Ignoring the issues related to mapping class groups, I am interested in certain representations of surface groups into Lie groups because of the so-called Anosov structures they define. These Anosov structures are a linearized version of Knneth structures studied in other contexts, a connection that has not been made before.

A.9 Lawton, Sean

The fundamental group of a surface with non-empty boundary is a free group. Consequently, the surface group representations into a (reductive) linear algebraic group are a smooth variety. A natural equivalence placed on representations is that of conjugacy. For instance conjugacy classes of representations parameterize bundles over a surface. Unfortunately, the orbit space of conjugacy classes is often difficult to analyze (not even Hausdorff). However, there is a closely related space known as the character variety. It is an algebraic quotient arising from extending the equivalences in the orbit space just enough to make
the resulting space a variety. In practice, the variety parameterizes representations that are sums of irreducibles; that is, the completely reducible representations. As it is a variety, one naturally wants to understand its ideal of defining relations.

It is here where there are open problems abound. In fact, a complete understanding is only known for two by two matrices (see Drensky). It was Procesi that summed up the situation best when, in his seminal paper describing sufficient generators and how the relations arise, he suggested that a complete description of the invariants was presently outside the theory. This holds true enough thirty years later as well. A concrete problem is to describe the ideal of relations defining the character variety of a rank 3 free group into $SL(3)$—this is an involved computational problem to say the least since the variety is 16 dimensional but yet there are 45 minimal generators.

It would be worth spending time exploring how additional structures, coming from geometry, could possibly aid in the understanding of the algebraic structure of these varieties by hopefully providing symmetries to reduce the computational complexity involved. In particular, many approaches (analytic, dynamic, graph theoretic, etc.) have been successfully employed in understanding aspects of these spaces (components, homotopy types, singularities, symmetries, foliations, etc.).

On the other hand, one would hope as further information is determined about the explicit algebraic structure, that one should be able to understand deformations of special classes of these representations, such as those with discrete image or those which preserve torsion (perhaps leading to coordinates on components). However, speculation that this ought to be true and doing it are very different—it would be worth exploring how to realize this with known examples.

Additionally, it would be interesting to explore possible connections between a possible generalization of the dynamics in the case of representations into $SL(2)$ to representations into $SL(3)$, and the algebraic symmetries in the latter case coming from the the outer automorphisms of the surface group.

**A.10 Link, Gabriele**

I am interested in the study of discrete subgroups of real semisimple Lie groups and their action on the geometric boundary of higher rank symmetric spaces of the noncompact type. The surface group representations in real semisimple Lie groups with maximal Toledo invariant provide an interesting concrete example of such groups. I intend to study of the limit set of these groups in the geometric boundary of the associated Hermitian symmetric space which hopefully will lead to a better understanding of surface group actions in certain components of the moduli space of representations. I am very keen on learning about the various different approaches which have been used so far in the study of surface group representations. Moreover I would like to discuss and exchange ideas with other participants of the conference concerning the geometric action of these surface group representations on the associated Hermitian symmetric spaces.

**A.11 Melnick, Karin**

I am interested in actions of surface groups and free groups on the 3-dimensional Einstein space, $Ein^3$, that are Kleinian—that is, that are properly discontinuous on an open subset. The 3-dimensional Einstein space is the conformal boundary of the constant-negative-curvature Lorentz manifold, anti-de Sitter space, of dimension 4, and free Kleinian actions
on it give rise to 3-dimensional conformally flat Lorentz manifolds. $Ein^3$ is also a parabolic quotient $G/P$ where $G = O(2, 3)$, and conformal actions on $Ein^3$ are given by representations in $G$. Finally, Einstein space is the Shilov boundary of the Hermitian symmetric space associated to $Sp(4, R)$ ($Sp(4, R)$ is locally isomorphic to $O(2, 3)$); the Shilov boundary is the unique closed $G$-orbit in the visual boundary of the symmetric space of $G$.

A.12 Mondello, Gabriele

I have been interested in Weil-Petersson geometry of the Teichmüller space of surfaces with boundary and its relation to Penner-Luo-Kontsevich coordinates.

I would like to understand how these coordinates extends to other Lie groups than $SL_2(\mathbb{R})$ and how to represent the natural symplectic, Riemannian and holomorphic structures in these coordinates.

I would also like to learn more about Higgs bundle techniques, harmonic maps and application of Morse theory to cohomology of these spaces.

A.13 Tan, Ser

Together with my co-workers, we are interested in the space of representations of surface groups into $G := SL(2, \mathbb{C})$, (and various subgroups of $G$); as well as in the dynamics of the action of the mapping class group on this space. In the simplest non-trivial case, the surface is a one-holed torus, its fundamental group $\pi$ is free on two generators and the mapping class group $\Gamma$ is isomorphic to $PSL(2, \mathbb{Z})$. Even in this case, the dynamics of the mapping class group action is already highly non-trivial and mysterious, and although various results have been obtained by Goldman for real $SL(2)$ representations, and by Goldman and Stantchev for “imaginary” representations, the general picture for $SL(2, \mathbb{C})$ representations is far from clear. In particular, together with my co-workers, we give conditions (Bowditch conditions) for describing the largest open subset $X_B$ of the character variety $X := Hom(\pi, G)//G$ on which $\Gamma$ acts properly discontinuously. As these conditions are algorithmically verifiable, one can obtain computer generated pictures of these sets which indicate their complexity and highly fractal like nature. We can also prove “McShane”-type identities for characters in this set, as well as certain characters on the boundary of this set, with applications to hyperbolic three manifolds which are obtained by hyperbolic Dehn surgery on a punctured torus bundle over the circle. It would be interesting to find applications of these identities to the study of 3-manifolds, and also to understand the connection with hyperbolic Dehn surgery space, which is still unclear.

We also study the dynamics of the mapping class group action by giving a definition of end invariants for a character $\rho$ in the character variety (these are elements of the projective lamination space of the torus $T$, and the definition generalizes the definition of a geometrically infinite end in the case of discrete, faithful representations), and studying the set of end invariants for a general character $\rho$. These end invariants show to what extent the action is not proper. For real and imaginary characters, as well as characters with discrete simple length spectrum, we show that this set has a simple structure, namely, it is empty, has one or two elements, is a Cantor set, or is the entire projective lamination space. An open problem is if this classification is true for all characters. Another interesting problem is to study the boundary of the set $X_B$ as well as the complement of the set $X_B$ in $X$. An open question is whether the complement always has non-empty interior in the relative character varieties.
Finally, the extension of the above to general surface groups is still largely unexplored, outside of the context of Kleinian groups and quasi-fuchsian groups.

**A.14 Toledo, Domingo**

My main interest in this workshop is to learn more of the recent and ongoing work on the structure of various special classes of representations of surface groups: maximal representations, Hitchin representations. It would be nice to see more direct relations among the approaches being used, such as bounded cohomology, Anosov structures, harmonic maps, Higgs bundles. I am also interested in representations of other groups, for instance fundamental groups of higher dimensional real or complex hyperbolic manifolds.

**A.15 Vakil, Ravi**

I'm interested in the moduli space of curves from an algebro-geometric point of view, so what I'm looking for in this workshop is a chance to understand current work (and open questions) in the theory of representations of surface groups (which is central to many aspects of curve theory), and to what extent I can find either interesting problems that can be addressed algebro-geometrically, or interesting methods that can be used to address algebro-geometric problems.

**A.16 Wienhard, Anna**

I am interested in various questions about representations of finitely generated groups into semisimple Lie groups, involving rigidity phenomena as well as interesting moduli spaces of geometric structures. I like the interplay of methods from quite different fields of mathematics which play a role when studying surface groups and I want to get a better understanding of the relations and connections.

**A.17 Wilkin, Graeme**

My research centres around studying the topology of hyperkähler quotients, such as moduli spaces of semistable Higgs bundles over a compact Riemann surface and quiver varieties. Recently, with Georgios Daskalopoulos and Jonathan Weitsman, we used Morse theoretic methods in the spirit of Atiyah and Botts approach for holomorphic bundles to compute cohomological invariants for moduli spaces of rank 2 Higgs bundles, giving new results about the topology of the $SL(2, \mathbb{C})$ character variety of a Riemann surface.

At this workshop I hope communicate these results to other researchers in the field, and to learn more about other techniques that people are using to study representation spaces of surface groups, and what kinds of problems people are interested in.

**A.18 Wolf, Michael**

I have been interested for some time in harmonic maps between surfaces, and in how the deformations of these maps reflect the Teichmüller theory of the domain or range surfaces. I hope to understand the relationship of this area to the work of others at the conference.

As a secondary interest, I have recently done some work on grafting of surfaces, and I’d hope to better understand the relationship of that operation and map to the main subject of the conference.