

Proof of Corollary 1

Summary of the proof

All three residues

In[257]:= **Simplify**[3 Residue[Residue[Residue[f1[s, w1, w2], {s, 1}], {w2, 0}], {w1, 0}] -
3 Residue[Residue[Residue[f2[s, w1, w2], {s, 1}], {w2, 1}], {w1, 0}] +
Residue[Residue[Residue[f3[s, w1, w2], {s, 1}], {w2, 1}], {w1, 1}]]

Out[257]:= X (0.287236647746619417221664617814645950166036274397222249618913907447198 +
0.677863310832980388541571083062733656003222322704135348688102425159897 Log[X] +
2.02119605787987777943324240784753809467091508369917789267040603543881 Log[X]^2 +
0.710113929053644747553958926673505372958197119463757504939845715359739 Log[X]^3 +
0.0544446791548840945807518785298617032826994387503389844120691008809066 Log[X]^4)

Summary of the constants

In[174]:= f1[s_, w1_, w2_] := X^((w1 + 2 w2 + 3 s) / 3) / w1 / w2 / s *
Zeta[s]^3 * Zeta[w1 + w2 + 1] * Zeta[w2 + 1] * A1[s, w1, w2];
f2[s_, w1_, w2_] := X^((w1 + 2 w2 + s) / 3) / w1 / w2 / s *
Zeta[s]^3 * Zeta[w1 + w2 + 1 - s] * Zeta[w2 + 1 - s] * A2[s, w1, w2];
f3[s_, w1_, w2_] := X^((w1 + w2 + s) / 3) / w1 / w2 / s *
Zeta[s]^3 * Zeta[w1 + 1 - s] * Zeta[w2 + 1 - s] * A3[s, w1, w2];

A1^(0,0,1)[1, 0, 0] := A1[1, 0, 0] * A111^(0,0,1)[1, 0, 0];
A1^(0,1,0)[1, 0, 0] := A1[1, 0, 0] * A111^(0,1,0)[1, 0, 0];
A1^(1,0,0)[1, 0, 0] := A1[1, 0, 0] * A111^(1,0,0)[1, 0, 0];
A1^(0,1,1)[1, 0, 0] :=
A1[1, 0, 0] A111^(0,0,1)[1, 0, 0] A111^(0,1,0)[1, 0, 0] + A1[1, 0, 0] A111^(0,1,1)[1, 0, 0];
A1^(1,1,1)[1, 0, 0] := A1[1, 0, 0] A111^(0,0,1)[1, 0, 0] A111^(0,1,0)[1, 0, 0] A111^(1,0,0)[1, 0, 0] +
A1[1, 0, 0] A111^(0,1,1)[1, 0, 0] A111^(1,0,0)[1, 0, 0] +
A1[1, 0, 0] A111^(0,1,0)[1, 0, 0] A111^(1,0,1)[1, 0, 0] +
A1[1, 0, 0] A111^(0,0,1)[1, 0, 0] A111^(1,1,0)[1, 0, 0] + A1[1, 0, 0] A111^(1,1,1)[1, 0, 0];
A1^(0,0,2)[1, 0, 0] := A1[1, 0, 0] A111^(0,0,1)[1, 0, 0]^2 + A1[1, 0, 0] A111^(0,0,2)[1, 0, 0];
A1^(0,2,0)[1, 0, 0] := A1[1, 0, 0] A111^(0,1,0)[1, 0, 0]^2 + A1[1, 0, 0] A111^(0,2,0)[1, 0, 0];
A1^(1,0,1)[1, 0, 0] :=
A1[1, 0, 0] A111^(0,0,1)[1, 0, 0] A111^(1,0,0)[1, 0, 0] + A1[1, 0, 0] A111^(1,0,1)[1, 0, 0];
A1^(1,1,0)[1, 0, 0] :=

$$\begin{aligned}
& A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(1,1,0)}[1, 0, 0]; \\
A1^{(1,2,0)}[1, 0, 0] & := A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0]^2 A111^{(1,0,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,2,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,1,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(1,2,0)}[1, 0, 0]; \\
A1^{(2,0,0)}[1, 0, 0] & := A1[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + A1[1, 0, 0] A111^{(2,0,0)}[1, 0, 0]; \\
A1^{(2,0,1)}[1, 0, 0] & := A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + \\
& 2 A1[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,0,1)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(2,0,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(2,0,1)}[1, 0, 0]; \\
A1^{(2,1,0)}[1, 0, 0] & := A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + \\
& 2 A1[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,1,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(2,0,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(2,1,0)}[1, 0, 0]; \\
A1^{(2,1,1)}[1, 0, 0] & := A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + \\
& A1[1, 0, 0] A111^{(0,1,1)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + \\
& 2 A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,0,1)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,1,0)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(1,0,1)}[1, 0, 0] A111^{(1,1,0)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,1,1)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(2,0,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,1,1)}[1, 0, 0] A111^{(2,0,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(2,0,1)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,0,1)}[1, 0, 0] A111^{(2,1,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(2,1,1)}[1, 0, 0]; \\
A1^{(2,2,0)}[1, 0, 0] & := A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0]^2 A111^{(1,0,0)}[1, 0, 0]^2 + \\
& A1[1, 0, 0] A111^{(0,2,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0]^2 + \\
& 4 A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,1,0)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(1,1,0)}[1, 0, 0]^2 + 2 A1[1, 0, 0] A111^{(1,0,0)}[1, 0, 0] A111^{(1,2,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0]^2 A111^{(2,0,0)}[1, 0, 0] + \\
& A1[1, 0, 0] A111^{(0,2,0)}[1, 0, 0] A111^{(2,0,0)}[1, 0, 0] + \\
& 2 A1[1, 0, 0] A111^{(0,1,0)}[1, 0, 0] A111^{(2,1,0)}[1, 0, 0] + A1[1, 0, 0] A111^{(2,2,0)}[1, 0, 0];
\end{aligned}$$

$$\begin{aligned}
A2^{(0,0,1)}[1, 0, 1] & := A2[1, 0, 1] * A222^{(0,0,1)}[1, 0, 1]; \\
A2^{(0,1,0)}[1, 0, 1] & := A2[1, 0, 1] * A222^{(0,1,0)}[1, 0, 1]; \\
A2^{(1,0,0)}[1, 0, 1] & := A2[1, 0, 1] * A222^{(1,0,0)}[1, 0, 1]; \\
A2^{(0,1,1)}[1, 0, 1] & := \\
& A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] + A2[1, 0, 1] A222^{(0,1,1)}[1, 0, 1]; \\
A2^{(0,1,2)}[1, 0, 1] & := A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1]^2 A222^{(0,1,0)}[1, 0, 1] + \\
& A2[1, 0, 1] A222^{(0,0,2)}[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] + \\
& 2 A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1] A222^{(0,1,1)}[1, 0, 1] + A2[1, 0, 1] A222^{(0,1,2)}[1, 0, 1]; \\
A2^{(0,0,2)}[1, 0, 1] & := A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1]^2 + A2[1, 0, 1] A222^{(0,0,2)}[1, 0, 1]; \\
A2^{(1,0,1)}[1, 0, 1] & := \\
& A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1] A222^{(1,0,0)}[1, 0, 1] + A2[1, 0, 1] A222^{(1,0,1)}[1, 0, 1]; \\
A2^{(1,1,0)}[1, 0, 1] & := \\
& A2[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] A222^{(1,0,0)}[1, 0, 1] + A2[1, 0, 1] A222^{(1,1,0)}[1, 0, 1]; \\
A2^{(1,1,1)}[1, 0, 1] & := A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] A222^{(1,0,0)}[1, 0, 1] + \\
& A2[1, 0, 1] A222^{(0,1,1)}[1, 0, 1] A222^{(1,0,0)}[1, 0, 1] +
\end{aligned}$$

$$\begin{aligned}
& A2[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] A222^{(1,0,1)}[1, 0, 1] + \\
& A2[1, 0, 1] A222^{(0,0,1)}[1, 0, 1] A222^{(1,1,0)}[1, 0, 1] + A2[1, 0, 1] A222^{(1,1,1)}[1, 0, 1]; \\
A2^{(2,0,0)}[1, 0, 1] & := A2[1, 0, 1] A222^{(1,0,0)}[1, 0, 1]^2 + A2[1, 0, 1] A222^{(2,0,0)}[1, 0, 1]; \\
A2^{(2,1,0)}[1, 0, 1] & := A2[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] A222^{(1,0,0)}[1, 0, 1]^2 + \\
& 2 A2[1, 0, 1] A222^{(1,0,0)}[1, 0, 1] A222^{(1,1,0)}[1, 0, 1] + \\
& A2[1, 0, 1] A222^{(0,1,0)}[1, 0, 1] A222^{(2,0,0)}[1, 0, 1] + A2[1, 0, 1] A222^{(2,1,0)}[1, 0, 1];
\end{aligned}$$

$$\begin{aligned}
A3^{(0,0,1)}[1, 1, 1] & := A3[1, 1, 1] * A333^{(0,0,1)}[1, 1, 1]; \\
A3^{(0,1,0)}[1, 1, 1] & := A3[1, 1, 1] * A333^{(0,1,0)}[1, 1, 1]; \\
A3^{(1,0,0)}[1, 1, 1] & := A3[1, 1, 1] * A333^{(1,0,0)}[1, 1, 1]; \\
A3^{(0,0,2)}[1, 1, 1] & := A3[1, 1, 1] A333^{(0,0,1)}[1, 1, 1]^2 + A3[1, 1, 1] A333^{(0,0,2)}[1, 1, 1]; \\
A3^{(0,1,1)}[1, 1, 1] & := \\
& A3[1, 1, 1] A333^{(0,0,1)}[1, 1, 1] A333^{(0,1,0)}[1, 1, 1] + A3[1, 1, 1] A333^{(0,1,1)}[1, 1, 1]; \\
A3^{(0,2,0)}[1, 1, 1] & := A3[1, 1, 1] A333^{(0,1,0)}[1, 1, 1]^2 + A3[1, 1, 1] A333^{(0,2,0)}[1, 1, 1]; \\
A3^{(1,0,1)}[1, 1, 1] & := \\
& A3[1, 1, 1] A333^{(0,0,1)}[1, 1, 1] A333^{(1,0,0)}[1, 1, 1] + A3[1, 1, 1] A333^{(1,0,1)}[1, 1, 1]; \\
A3^{(1,1,0)}[1, 1, 1] & := \\
& A3[1, 1, 1] A333^{(0,1,0)}[1, 1, 1] A333^{(1,0,0)}[1, 1, 1] + A3[1, 1, 1] A333^{(1,1,0)}[1, 1, 1]; \\
A3^{(2,0,0)}[1, 1, 1] & := A3[1, 1, 1] A333^{(1,0,0)}[1, 1, 1]^2 + A3[1, 1, 1] A333^{(2,0,0)}[1, 1, 1];
\end{aligned}$$

$$\begin{aligned}
A1[1, 0, 0] & := \\
& 0.217778716619536378323007514119446813130797755001355937648276403523626491112252620557\backslash \\
& 9254438235637656918339357748032\`100.;
\end{aligned}$$

$$A2[1, 0, 1] := A1[1, 0, 0];$$

$$A3[1, 1, 1] := A1[1, 0, 0];$$

$$\begin{aligned}
A111^{(0,1,0)}[1, 0, 0] & := \\
& 0.8430220578603099764198623764339648641000307466481332537296641875671192668876215912415\backslash \\
& 9165565392\`75.
\end{aligned}$$

$$A111^{(0,0,1)}[1, 0, 0] := 2 * A111^{(0,1,0)}[1, 0, 0];$$

$$A111^{(0,1,1)}[1, 0, 0] :=$$

$$\begin{aligned}
& -1.190727816059283072434053614108579419745932058046617974308297869696307135808149094290\backslash \\
& 370550743993\`75.
\end{aligned}$$

$$A111^{(0,2,0)}[1, 0, 0] :=$$

$$\begin{aligned}
& -1.45020908785524954082887708718731238277383605148758863124660826169394362046555887276\backslash \\
& 9919442550403\`75.;
\end{aligned}$$

$$A111^{(1,0,0)}[1, 0, 0] := 3 * A111^{(0,1,0)}[1, 0, 0];$$

$$A111^{(1,0,1)}[1, 0, 0] :=$$

$$\begin{aligned}
& -4.326149391201720525313021068795706331429266638208531355130930119053118430049057281918\backslash \\
& 885962161142\`75.
\end{aligned}$$

$$A111^{(1,1,1)}[1, 0, 0] :=$$

$$\begin{aligned}
& 4.922546395398470985919121032181312118733181980067924933193051055404874059727834172602\backslash \\
& 287213077508\`75.;
\end{aligned}$$

$$A111^{(1,0,2)}[1, 0, 0] := 2 * A111^{(1,1,1)}[1, 0, 0];$$

$$A111^{(1,1,0)}[1, 0, 0] := A111^{(1,0,1)}[1, 0, 0] / 2;$$

$A111^{(1,2,0)} [1, 0, 0] :=$
 5. 203442844714798964018612580621705162127001410139188853357799078660730898691806653152\;
 754725469967`75.;

$A111^{(2,0,0)} [1, 0, 0] :=$
 -2. 79373963278994981211769042308953937015408419381694195210996243309601195345221795508\;
 1858689463119`75.;

$A111^{(2,1,0)} [1, 0, 0] :=$
 4. 641649946082143007819629483740919075339362549996661013028303032149017220763861683754\;
 510494643289`75.;

$A111^{(2,0,1)} [1, 0, 0] := 2 * A111^{(2,1,0)} [1, 0, 0];$
 $A111^{(0,0,2)} [1, 0, 0] := 2 * A111^{(0,1,1)} [1, 0, 0];$
 $A111^{(2,1,1)} [1, 0, 0] :=$
 -16. 66209490925358764598162624603931187964586323483510444961358657708151680208949692375\;
 4264885014504`75.

$A111^{(2,2,0)} [1, 0, 0] :=$
 -16. 1369857125556344357955245341546849960669129554600129510664007388581442478420424490\;
 94068963096076`75.;

$A222^{(0,1,0)} [1, 0, 1] := A111^{(0,1,0)} [1, 0, 0];$
 $A222^{(0,0,1)} [1, 0, 1] := 2 * A111^{(0,1,0)} [1, 0, 0];$
 $A222^{(1,0,0)} [1, 0, 1] := A111^{(0,1,0)} [1, 0, 0];$
 $A222^{(0,1,1)} [1, 0, 1] := A111^{(0,1,1)} [1, 0, 0];$
 $A222^{(0,0,2)} [1, 0, 1] := A111^{(0,0,2)} [1, 0, 0];$
 $A222^{(0,1,2)} [1, 0, 1] :=$
 2. 253473304856102095489192873464602805750241458417489508846347471733613639206760865244\;
 491245099618`75.;

$A222^{(1,0,1)} [1, 0, 1] := 2 * A222^{(1,1,0)} [1, 0, 1];$
 $A222^{(1,1,0)} [1, 0, 1] :=$
 -0. 97234687954157719022245692028927374596870126105764770325716718983025207921637954823\;
 1505572162349`75.;

$A222^{(2,0,0)} [1, 0, 1] :=$
 3. 477103517494925093640244486284714453212584966506884809535302065617610635029598419913\;
 002810379413`75.;

$A222^{(1,1,1)} [1, 0, 1] :=$
 2. 669073090542368890429928158716709312982940521650435424346703583671623039891572526148\;
 00129547396`75.;

$A222^{(2,1,0)} [1, 0, 1] :=$
 -2. 94996953985869686852941970715710235637675995172169934451145160692747987885554500889\;
 4380831862888`75.;

$A333^{(0,0,1)} [1, 1, 1] := A111^{(0,1,0)} [1, 0, 0];$
 $A333^{(0,1,0)} [1, 1, 1] := A333^{(0,0,1)} [1, 1, 1];$
 $A333^{(0,0,2)} [1, 1, 1] := A111^{(0,2,0)} [1, 0, 0];$
 $A333^{(0,2,0)} [1, 1, 1] := A111^{(0,2,0)} [1, 0, 0];$
 $A333^{(0,1,1)} [1, 1, 1] :=$
 0. 259481271795966468394823473078732963027903993440970656938310391997636484657409776724\;

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613359110604`75.;
A333(1,1,0)[1, 1, 1] := A222(1,1,0)[1, 0, 1];
A333(1,0,1)[1, 1, 1] := A333(1,1,0)[1, 1, 1];
A333(1,0,0)[1, 1, 1] := A111(0,1,0)[1, 0, 0];
A333(2,0,0)[1, 1, 1] := A222(2,0,0)[1, 0, 1];

```

Definitions

```

In[ ]:= A11[s_, w1_, w2_] := (1 - p-1-w2) (1 - p-1-w1-w2)
  (
    1 + 
$$\frac{(1 - p^{-s})^3 \left( \frac{1}{-1+p^{1-w2}} + \frac{1}{-1+p^{1+w1-w2}} + \frac{1}{(-1+p^{1-w2})(-1+p^{1+w1-w2})} \right)}{1 - \frac{1}{p}}$$

  ) // . p -> Prime[n];

A22[s_, w1_, w2_] := (1 - pS-1-w2) (1 - pS-1-w1-w2)
  (
    1 + 
$$\frac{(1 - p^{-s})^3 \left( \frac{1}{-1+p^{1-w2-s}} + \frac{1}{-1+p^{1+w1-w2-s}} + \frac{1}{(-1+p^{1-w2-s})(-1+p^{1+w1-w2-s})} \right)}{1 - \frac{1}{p}}$$

  ) // . p -> Prime[n];

A33[s_, w1_, w2_] := (1 - pS-1-w2) (1 - pS-1-w1)
  (
    1 + 
$$\frac{(1 - p^{-s})^3 \left( \frac{1}{-1+p^{1-w2-s}} + \frac{1}{-1+p^{1+w1-s}} + \frac{1}{(-1+p^{1-w2-s})(-1+p^{1+w1-s})} \right)}{1 - \frac{1}{p}}$$

  ) // . p -> Prime[n];

```

Computations of the constants

Constants from the first residue

```

A111(0,1,0)[1, 0, 0] :=
0.843022057860309976419862376433964864100030746648133253729664187567119266887621591241591.
65565392`75.;

```

```

In[ ]:= Simplify[D[Log[A11[s, w1, w2]], w1] // . {s -> 1, w1 -> 0, w2 -> 0}]

```

```

Out[ ]:= 
$$\frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$


```

```

In[ ]:= N[Sum[
$$\frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$
, {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000}, Do[CC = Join[{0},
Series[
$$\frac{(-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$
 // Prime[n] → 1/x, {x, 0, t}][[3]];
Print[N[-Sum[CC[[k]] * (PrimeZetaP'[k] + Log[2] / 2^k), {k, 1, Length[CC]}] +

$$\frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$
 // Prime[n] → 2, 75]], {t, 1000, 1500, 100}]]]

```

Out[]:= 0.8430029907

0.843022057860309976419862376433964864100030746648133253729664187567119266888

0.843022057860309976419862376433964864100030746648133253729664187567119266888

Out[]:= \$Aborted

```

In[ ]:= A111(0,1,0)[1, 0, 0] :=
0.843022057860309976419862376433964864100030746648133253729664187567119266887621591241\
59165565392`75.;

```

A111^(0,0,1)[1, 0, 0] := 2 * A111^(0,1,0)[1, 0, 0];

```

In[ ]:= Simplify[D[Log[A11[s, w1, w2]], w2] // {s → 1, w1 → 0, w2 → 0}]

```

Out[]:=
$$\frac{2 \text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$

```

In[ ]:= A111(0,0,1)[1, 0, 0] := 2 * A111(0,1,0)[1, 0, 0];

```

A111^(0,1,1)[1, 0, 0] :=
-1.1907278160592830724340536141085794197459320580466179743082978696963071358081490942903
70550743993`75.;

```

In[ ]:= Simplify[D[D[Log[A11[s, w1, w2]], w1], w2] // {s → 1, w1 → 0, w2 → 0}]

```

Out[]:=
$$-\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + \text{Prime}[n] + 2 \text{Prime}[n]^2)}{(-1 + \text{Prime}[n]) (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2)^2}$$

```

In[ ]:= N[Sum[- $\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + \text{Prime}[n] + 2 \text{Prime}[n]^2)}{(-1 + \text{Prime}[n]) (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2)^2}$ , {n, 1, 10^4}], 10]

Block[{$MaxExtraPrecision = 1000}, Do[CC = Join[{0}, Series[

$$-\frac{\text{Prime}[n] (-1 + \text{Prime}[n] + 2 \text{Prime}[n]^2)}{(-1 + \text{Prime}[n]) (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2)^2}$$
 // Prime[n] → 1/x, {x, 0, t}][[3]]];

Print[
N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2/2^k - Log[3]^2/3^k), {k, 1, Length[CC]}] -

$$\frac{18 \text{Log}[2]^2}{49} - \frac{15 \text{Log}[3]^2}{98}$$
, 75]], {t, 1000, 1200, 50}]]

```

Out[]:= -1.190488334

-1.19072781605928307243405361410857941974593205804661797430829786969630713581

-1.19072781605928307243405361410857941974593205804661797430829786969630713581

-1.19072781605928307243405361410857941974593205804661797430829786969630713581

Out[]:= \$Aborted

A111^(0,1,1)[1, 0, 0] :=

**-1.19072781605928307243405361410857941974593205804661797430829786969630713580814909429;
0370550743993`75.;**

A111^(0,2,0)[1, 0, 0] :=

**-1.4502090878552495408288770871873123827738360514875886312466082616939436204655588727699;
19442550403`75.;**

In[]:= Simplify[D[D[Log[A11[s, w1, w2]], w1], w1] // {s → 1, w1 → 0, w2 → 0}

```

Out[ ]:= 
$$-\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$



$$-\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$


```

```
in[ ]:= N[Sum[- $\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$ , {n, 1, 10^5}], 10]
```

```
Block[{$MaxExtraPrecision = 1000},
```

```
Do[CC = Join[{0}, Series[- $\frac{\text{Prime}[n] (1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$  //.
```

```
Prime[n] → 1/x, {x, 0, t}][[3]]];
```

```
Print[N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2/2^k), {k, 1, Length[CC]}] -
```

```
 $\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$  //.
```

```
n → 1, 75]], {t, 950, 1000, 25}]]]
```

```
Out[ ]:= -1.450185894
```

```
-1.45020908785524954082887708718731238277383605148758863124660826169394362047
```

```
-1.45020908785524954082887708718731238277383605148758863124660826169394362047
```

```
-1.45020908785524954082887708718731238277383605148758863124660826169394362047
```

```
A111(0,2,0)[1, 0, 0] :=
```

```
-1.4502090878552495408288770871873123827738360514875886312466082616939436204655588727639919442550403`75.;
```

```
A111(1,0,0)[1, 0, 0] := 3 * A111(0,1,0)[1, 0, 0];
```

```
in[ ]:= Simplify[D[Log[A11[s, w1, w2]], s] // {s → 1, w1 → 0, w2 → 0}]
```

```
Out[ ]:=  $\frac{3 \text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$ 
```

```
in[ ]:= A111(1,0,0)[1, 0, 0] := 3 * A111(0,1,0)[1, 0, 0];
```

```
A111(1,0,1)[1, 0, 0] :=
```

```
-4.3261493912017205253130210687957063314292666382085313551309301190531184300490572819188385962161142`75.;
```

```
in[ ]:= Simplify[D[D[Log[A11[s, w1, w2]], s], w2] // {s → 1, w1 → 0, w2 → 0}]
```

```
Out[ ]:=  $-\frac{6 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n]^4}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$ 
```



```

In[ ]:= N[Sum[- $\frac{6 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n]^4}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$ , {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000}, Do[CC = Join[{0},
Series[- $\frac{6 \text{Prime}[n]^4}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$  // Prime[n] → 1/x, {x, 0, t}][[3]];
Print[N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] -
 $\frac{6 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n]^4}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$  // Prime[n] → 2, 75]], {t, 1000, 1100, 50}]]]

```

Out[]:= -4. 325430944

-4. 32614939120172052531302106879570633142926663820853135513093011905311843005

-4. 32614939120172052531302106879570633142926663820853135513093011905311843005

-4. 32614939120172052531302106879570633142926663820853135513093011905311843005

A111^(1,0,1)[1, 0, 0] :=

**-4. 32614939120172052531302106879570633142926663820853135513093011905311843004905728191:
8885962161142`75. ;**

A111^(1,1,1)[1, 0, 0] :=

**4. 922546395398470985919121032181312118733181980067924933193051055404874059727834172602287:
213077508`75. ;**

```

In[ ]:= Simplify[D[D[D[Log[A11[s, w1, w2]], s], w1], w2] // {s → 1, w1 → 0, w2 → 0}

```

$3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)$

```

Out[ ]:=  $\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$ 

```

```

In[ ]:= N[Sum[ $\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$ , {n, 1, 10^4}], 10]

```

```

Block[{$MaxExtraPrecision = 1250}, Do[CC = Join[{0},

```

```

Series[ $\frac{3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$  // Prime[n] → 1/x, {x, 0, t}][[3]];

```

```

Print[N[-Sum[CC[[k]] * (PrimeZetaP'''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +

```

```

 $\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$  //

```

```

Prime[n] → 2, 75]], {t, 1000, 1100, 50}]]]

```

Out[]:= 4. 918005965

4. 92254639539847098591912103218131211873318198006792493319305105540487405973
 4. 92254639539847098591912103218131211873318198006792493319305105540487405973
 4. 92254639539847098591912103218131211873318198006792493319305105540487405973

A111^(1,1,1) [1, 0, 0] :=

**4. 922546395398470985919121032181312118733181980067924933193051055404874059727834172602.
 287213077508`75. ;**

A111^(1,0,2) [1, 0, 0] := 2 * A111^(1,1,1) [1, 0, 0] ;

In[]:= Simplify[D[D[D[Log[A11[s, w1, w2]], s], w2], w2] /. {s -> 1, w1 -> 0, w2 -> 0}]

$$\text{Out[]:= } \frac{6 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-2 + 3 \text{Prime}[n] + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$

*In[]:= A111^(1,0,2) [1, 0, 0] := 2 * A111^(1,1,1) [1, 0, 0] ;*

A111^(1,1,0) [1, 0, 0] := A111^(1,0,1) [1, 0, 0] / 2 ;

In[]:= Simplify[D[D[Log[A11[s, w1, w2]], s], w1] /. {s -> 1, w1 -> 0, w2 -> 0}]

$$\text{Out[]:= } - \frac{3 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n]^4}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$

In[]:= A111^(1,1,0) [1, 0, 0] := A111^(1,0,1) [1, 0, 0] / 2 ;

A111^(1,2,0) [1, 0, 0] :=

**5. 203442844714798964018612580621705162127001410139188853357799078660730898691806653152754.
 725469967`75. ;**

In[]:= Simplify[D[D[D[Log[A11[s, w1, w2]], s], w1], w1] /. {s -> 1, w1 -> 0, w2 -> 0}]

$$\text{Out[]:= } \frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-1 + \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$

```

In[ ]:= N[Sum[
$$\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-1 + \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
, {n, 1, 10^4}], 10]

```

```

Block[{$MaxExtraPrecision = 1250},

```

```

Do[CC = Join[{0}, Series[
$$\frac{3 \text{Prime}[n]^4 (-1 + \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
 // .

```

```

Prime[n] → 1 / x, {x, 0, t}][[3]];

```

```

Print[N[-Sum[CC[[k]] * (PrimeZetaP''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +

```

```


$$\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-1 + \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
 // .

```

```

Prime[n] → 2, 75]], {t, 1000, 1100, 50}]]]

```

```

Out[ ]:= 5.198902394

```

```

5.20344284471479896401861258062170516212700141013918885335779907866073089869

```

```

5.20344284471479896401861258062170516212700141013918885335779907866073089869

```

```

5.20344284471479896401861258062170516212700141013918885335779907866073089869

```

```

In[ ]:= A111(1,2,0)[1, 0, 0] :=

```

```

5.203442844714798964018612580621705162127001410139188853357799078660730898691806653152.;
754725469967^75.;

```

```

A111(2,0,0)[1, 0, 0] :=

```

```

-2.7937396327899498121176904230895393701540841938169419521099624330960119534522179550818.;
58689463119^75.;

```

```

In[ ]:= Simplify[D[D[Log[A11[s, w1, w2]], s], s] // . {s → 1, w1 → 0, w2 → 0}

```

```

Out[ ]:= -
$$\frac{3 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n]) (-1 - \text{Prime}[n] + \text{Prime}[n]^2)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$


```

```

In[ ]:= N[Sum[-
$$\frac{3 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n]) (-1 - \text{Prime}[n] + \text{Prime}[n]^2)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
, {n, 1, 10^4}],
10]
Block[{$MaxExtraPrecision = 1250},
Do[CC = Join[{0}, Series[-
$$\frac{3 \text{Prime}[n] (-1 + 2 \text{Prime}[n]) (-1 - \text{Prime}[n] + \text{Prime}[n]^2)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
 // .
Prime[n] -> 1/x, {x, 0, t}][[3]]];
Print[N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] -

$$\frac{3 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n]) (-1 - \text{Prime}[n] + \text{Prime}[n]^2)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
 // .
Prime[n] -> 2, 75]], {t, 1000, 1100, 50}]]

```

Out[]:= -2.793021191

-2.79373963278994981211769042308953937015408419381694195210996243309601195345

-2.79373963278994981211769042308953937015408419381694195210996243309601195345

-2.79373963278994981211769042308953937015408419381694195210996243309601195345

A111^(2,0,0)[1, 0, 0] :=

**-2.79373963278994981211769042308953937015408419381694195210996243309601195345221795508:
1858689463119^75.;**

A111^(2,1,0)[1, 0, 0] :=

**4.641649946082143007819629483740919075339362549996661013028303032149017220763861683754510:
494643289^75.;**

In[]:= **Simplify[D[D[D[Log[A11[s, w1, w2]], s], s], w1] // . {s -> 1, w1 -> 0, w2 -> 0}]**

Out[]:=
$$\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-3 + 5 \text{Prime}[n] - \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$

```

In[ ]:= N[Sum[
$$\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-3 + 5 \text{Prime}[n] - \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
, {n, 1, 10^4}], 10]

```

```

Block[{$MaxExtraPrecision = 1250},

```

```

Do[CC = Join[{0}, Series[
$$\frac{3 \text{Prime}[n]^4 (-3 + 5 \text{Prime}[n] - \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
 //.
```

```

Prime[n] → 1 / x, {x, 0, t}][[3]]];

```

```

Print[N[-Sum[CC[[k]] * (PrimeZetaP'''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +

```

```


$$\frac{3 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-3 + 5 \text{Prime}[n] - \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$
 //.
```

```

Prime[n] → 2, 75]], {t, 1000, 1100, 50}]]]

```

```

Out[ ]:= 4.637109535

```

```

4.64164994608214300781962948374091907533936254999666101302830303214901722076

```

```

4.64164994608214300781962948374091907533936254999666101302830303214901722076

```

```

4.64164994608214300781962948374091907533936254999666101302830303214901722076

```

```

In[ ]:= A111(2,1,0)[1, 0, 0] :=

```

```

4.641649946082143007819629483740919075339362549996661013028303032149017220763861683754.
510494643289`75.;

```

```

A111(2,0,1)[1, 0, 0] := 2 * A111(2,1,0)[1, 0, 0];

```

```

In[ ]:= Simplify[D[D[D[Log[A11[s, w1, w2]], s], s], w2] //.{s → 1, w1 → 0, w2 → 0}

```

```

Out[ ]:= 
$$\frac{6 \text{Log}[\text{Prime}[n]]^3 \text{Prime}[n]^4 (-3 + 5 \text{Prime}[n] - \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$


```

```

In[ ]:= A111(2,0,1)[1, 0, 0] := 2 * A111(2,1,0)[1, 0, 0];

```

```

A111(0,0,2)[1, 0, 0] := 2 * A111(0,1,1)[1, 0, 0];

```

```

In[ ]:= Simplify[D[D[Log[A11[s, w1, w2]], w2], w2] //.{s → 1, w1 → 0, w2 → 0}

```

```

Out[ ]:= 
$$-\frac{2 \text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + \text{Prime}[n] + 2 \text{Prime}[n]^2)}{(-1 + \text{Prime}[n]) (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2)^2}$$


```

```

In[ ]:= A111(0,0,2)[1, 0, 0] := 2 * A111(0,1,1)[1, 0, 0];

```

A111^(2,1,1)[1, 0, 0] :=

-16. 662094909253587645981626246039311879645863234835104449613586577081516802089496923754
264885014504`75. ;

In[]:= **Simpl i fy**[D[D[D[D[Log[A11[s, w1, w2]], s], s], w1], w2] //. {s → 1, w1 → 0, w2 → 0}]

Out[]:=
$$-\left(3 \operatorname{Log}[\operatorname{Prime}[n]]^4 \operatorname{Prime}[n]^4 \left(6 - 19 \operatorname{Prime}[n] + 17 \operatorname{Prime}[n]^2 - 8 \operatorname{Prime}[n]^3 + 20 \operatorname{Prime}[n]^4 - \operatorname{Prime}[n]^5 + \operatorname{Prime}[n]^6\right)\right) / \left(1 - 3 \operatorname{Prime}[n] + \operatorname{Prime}[n]^2 + \operatorname{Prime}[n]^3\right)^4$$

In[]:= **N**[Sum[-((3 Log[Prime[n]]⁴ Prime[n]⁴ (6 - 19 Prime[n] + 17 Prime[n]² - 8 Prime[n]³ + 20 Prime[n]⁴ - Prime[n]⁵ + Prime[n]⁶)) / (1 - 3 Prime[n] + Prime[n]² + Prime[n]³)⁴, {n, 1, 10⁴}], 10]

Block[\$MaxExtraPrecision = 1500],

Do[CC = Join[{0}, Series[-((3 Prime[n]⁴ (6 - 19 Prime[n] + 17 Prime[n]² - 8 Prime[n]³ + 20 Prime[n]⁴ - Prime[n]⁵ + Prime[n]⁶)) / (1 - 3 Prime[n] + Prime[n]² + Prime[n]³)⁴ // Prime[n] → 1/x, {x, 0, t}][[3]]];

Print[N[Sum[CC[[k]] * (PrimeZetaP''''[k] - Log[2]^4 / 2^k), {k, 1, Length[CC]}] +

-((3 Log[Prime[n]]⁴ Prime[n]⁴ (6 - 19 Prime[n] + 17 Prime[n]² - 8 Prime[n]³ + 20 Prime[n]⁴ - Prime[n]⁵ + Prime[n]⁶)) /

(1 - 3 Prime[n] + Prime[n]² + Prime[n]³)⁴ // Prime[n] → 2, 75]], {t, 950, 975, 25}]]

Out[]:= -16. 60429171

-16. 6620949092535876459816262460393118796458632348351044496135865770815168021

-16. 6620949092535876459816262460393118796458632348351044496135865770815168021

In[]:= **A111^(2,1,1)[1, 0, 0] :=**

-16. 6620949092535876459816262460393118796458632348351044496135865770815168020894969237
54264885014504`75. ;

A111^(2,2,0)[1, 0, 0] :=

-16. 136985712555634435795524534154684996066912955460012951066400738858144247842042449094
068963096076`75. ;

In[]:= **Simpl i fy**[D[D[D[D[Log[A11[s, w1, w2]], s], s], w1], w1] //. {s → 1, w1 → 0, w2 → 0}]

Out[]:=
$$-\frac{3 \operatorname{Log}[\operatorname{Prime}[n]]^4 \operatorname{Prime}[n]^4 \left(3 - 8 \operatorname{Prime}[n] + 3 \operatorname{Prime}[n]^2 + 11 \operatorname{Prime}[n]^4 + \operatorname{Prime}[n]^6\right)}{\left(1 - 3 \operatorname{Prime}[n] + \operatorname{Prime}[n]^2 + \operatorname{Prime}[n]^3\right)^4}$$

```

In[ ]:= N[Sum[- $\frac{3 \text{Log}[\text{Prime}[n]]^4 \text{Prime}[n]^4 (3 - 8 \text{Prime}[n] + 3 \text{Prime}[n]^2 + 11 \text{Prime}[n]^4 + \text{Prime}[n]^6)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^4}$ ,
{n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1500}, Do[CC =
Join[{0}, Series[- $\frac{3 \text{Prime}[n]^4 (3 - 8 \text{Prime}[n] + 3 \text{Prime}[n]^2 + 11 \text{Prime}[n]^4 + \text{Prime}[n]^6)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^4}$  // .
Prime[n] -> 1/x, {x, 0, t}][[3]]];
Print[N[Sum[CC[[k]] * (PrimeZetaP''''[k] - Log[2]^4 / 2^k), {k, 1, Length[CC]}] +
 $\frac{3 \text{Log}[\text{Prime}[n]]^4 \text{Prime}[n]^4 (3 - 8 \text{Prime}[n] + 3 \text{Prime}[n]^2 + 11 \text{Prime}[n]^4 + \text{Prime}[n]^6)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^4}$  // .
Prime[n] -> 2, 75]], {t, 950, 975, 25}]]]

```

Out[]:= -16.07918227

-16.1369857125556344357955245341546849960669129554600129510664007388581442478

-16.1369857125556344357955245341546849960669129554600129510664007388581442478

```

In[ ]:= A111(2,2,0)[1, 0, 0] :=
-16.1369857125556344357955245341546849960669129554600129510664007388581442478420424490.
94068963096076`75.;

```

Computing the 1st residue

```

In[253]:= R1 :=
N[Simplify[3 Residue[Residue[Residue[f1[s, w1, w2], {s, 1}], {w2, 0}], {w1, 0}], 100];
R1

```

```

Out[254]:= X (0.2162405696294719794753079400767624606303203012696111959327915428237555 +
1.496610227225105051189903151682707817888540922477154169127093433402851 Log[X] +
2.868588234840808522441173283738349340977413007893458212431898204954788 Log[X]^2 +
0.819003287363412936715462683733228779523595996964435473763983917121552 Log[X]^3 +
0.05444467915488409458075187852986170328269943875033898441206910088090662 Log[X]^4)

```

Constants from the 2nd residue

$A222^{(0,1,0)}[1, 0, 1] := A111^{(0,1,0)}[1, 0, 0];$

```

In[ ]:= Simplify[D[Log[A22[s, w1, w2]], w1] // . {s -> 1, w1 -> 0, w2 -> 1}]

```

```

Out[ ]:=  $\frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$ 

```

$$\text{In[]:= A222}^{(0,1,0)} [1, 0, 1] := \text{A111}^{(0,1,0)} [1, 0, 0];$$

$$\text{A222}^{(0,0,1)} [1, 0, 1] := 2 * \text{A111}^{(0,1,0)} [1, 0, 0];$$

$$\text{In[]:= Simplify[D[Log[A22[s, w1, w2]], w2] /. {s -> 1, w2 -> 1, w1 -> 0}]$$

$$\text{Out[]:= } \frac{2 \text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$

$$\text{A222}^{(0,0,1)} [1, 0, 1] := 2 * \text{A111}^{(0,1,0)} [1, 0, 0];$$

$$\text{A222}^{(1,0,0)} [1, 0, 1] := \text{A111}^{(0,1,0)} [1, 0, 0];$$

$$\text{In[]:= Simplify[D[Log[A22[s, w1, w2]], s] /. {s -> 1, w1 -> 0, w2 -> 1}]$$

$$\text{Out[]:= } \frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$

$$\text{In[]:= A222}^{(1,0,0)} [1, 0, 1] := \text{A111}^{(0,1,0)} [1, 0, 0];$$

$$\text{A222}^{(0,1,1)} [1, 0, 1] := \text{A111}^{(0,1,1)} [1, 0, 0];$$

$$\text{In[]:= Simplify[D[D[Log[A22[s, w1, w2]], w1], w2] /. {s -> 1, w1 -> 0, w2 -> 1}]$$

$$\text{Out[]:= } - \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + \text{Prime}[n] + 2 \text{Prime}[n]^2)}{(-1 + \text{Prime}[n]) (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2)^2}$$

$$\text{In[]:= A222}^{(0,1,1)} [1, 0, 1] := \text{A111}^{(0,1,1)} [1, 0, 0];$$

$$\text{A222}^{(0,1,2)} [1, 0, 1] :=$$

$$2.253473304856102095489192873464602805750241458417489508846347471733613639206760865244491.245099618^75.;$$

$$\text{In[]:= Simplify[D[D[D[Log[A22[s, w1, w2]], w1], w2], w2] /. {s -> 1, w1 -> 0, w2 -> 1}]$$

$$\text{Out[]:= } \frac{(\text{Log}[\text{Prime}[n]]^3 \text{Prime}[n] (-1 + \text{Prime}[n] + 6 \text{Prime}[n]^2 - 6 \text{Prime}[n]^3 - 3 \text{Prime}[n]^4 - 3 \text{Prime}[n]^5 + 2 \text{Prime}[n]^6))}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^3}$$


```

In[ ]:= N[Sum[(Log[Prime[n]]^3 Prime[n]
  (-1 + Prime[n] + 6 Prime[n]^2 - 6 Prime[n]^3 - 3 Prime[n]^4 - 3 Prime[n]^5 + 2 Prime[n]^6)) /
  (1 - 3 Prime[n] + Prime[n]^2 + Prime[n]^3)^3, {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1250},
  Do[CC = Join[{0}, Series[(Prime[n] (-1 + Prime[n] + 6 Prime[n]^2 -
    6 Prime[n]^3 - 3 Prime[n]^4 - 3 Prime[n]^5 + 2 Prime[n]^6)) /
    (1 - 3 Prime[n] + Prime[n]^2 + Prime[n]^3)^3 /. Prime[n] -> 1/x, {x, 0, t}][[3]]];
  Print[N[-Sum[CC[[k]] * (PrimeZetaP''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +
    (Log[Prime[n]]^3 Prime[n]
    (-1 + Prime[n] + 6 Prime[n]^2 - 6 Prime[n]^3 - 3 Prime[n]^4 - 3 Prime[n]^5 + 2 Prime[n]^6)) /
    (1 - 3 Prime[n] + Prime[n]^2 + Prime[n]^3)^3 /. Prime[n] -> 2, 75}], {t, 750, 750, 50}]]

```

Out[]:= 2. 250446371

2. 25347330485610209548919287346460280575024145841748950884634747173361363921

```

In[ ]:= A222(0,1,2)[1, 0, 1] :=
  2. 253473304856102095489192873464602805750241458417489508846347471733613639206760865244.
  491245099618`75.;

```

A222^(0,0,2)[1, 0, 1] := A111^(0,0,2)[1, 0, 0];

```

In[ ]:= Simplify[D[D[Log[A22[s, w1, w2]], w2], w2] /. {s -> 1, w1 -> 0, w2 -> 1}]

```

Out[]:=
$$-\frac{2 \operatorname{Log}[\operatorname{Prime}[n]]^2 \operatorname{Prime}[n] (-1 + \operatorname{Prime}[n] + 2 \operatorname{Prime}[n]^2)}{(-1 + \operatorname{Prime}[n]) (-1 + 2 \operatorname{Prime}[n] + \operatorname{Prime}[n]^2)^2}$$

A222^(0,0,2)[1, 0, 1] := A111^(0,0,2)[1, 0, 0];

A222^(1,0,1)[1, 0, 1] := 2 * A222^(1,1,0)[1, 0, 1];

```

In[ ]:= Simplify[D[D[Log[A22[s, w1, w2]], s], w2] /. {s -> 1, w1 -> 0, w2 -> 1}]

```

Out[]:=
$$-\frac{2 \operatorname{Log}[\operatorname{Prime}[n]]^2 \operatorname{Prime}[n] (-1 + 2 \operatorname{Prime}[n] + \operatorname{Prime}[n]^2 + \operatorname{Prime}[n]^3)}{(1 - 3 \operatorname{Prime}[n] + \operatorname{Prime}[n]^2 + \operatorname{Prime}[n]^3)^2}$$

A222^(1,0,1)[1, 0, 1] := 2 * A222^(1,1,0)[1, 0, 1];

A222^(1,1,0)[1, 0, 1] :=

**-0. 9723468795415771902224569202892737459687012610576477032571671898302520792163795482315.
 05572162349`75.;**

```

In[ ]:= Simplify[D[D[Log[A22[s, w1, w2]], s], w1] /. {s -> 1, w1 -> 0, w2 -> 1}]

```

$$\text{In[]:= } - \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$

$$- \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2} \quad // . \text{Prime}[n] \rightarrow 2$$

$$\text{Out[]:= } - \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$

$$\text{Out[]:= } - \frac{30}{49} \text{Log}[2]^2$$

$$\text{In[]:= } \mathbf{N} \left[\text{Sum} \left[- \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}, \{n, 1, 10^4\} \right], 10 \right]$$

Block [{ \$MaxExtraPrecision = 1000 } ,

Do [**CC** = **Join** [{ 0 } , **Series** [- $\frac{\text{Prime}[n] (-1 + 2 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$ // .

Prime[n] $\rightarrow 1/x$, {x, 0, t}] [[3]] ;

Print [**N** [**Sum** [**CC** [[k]] * (**PrimeZetaP** [[k]] - **Log**[2] ^ 2 / 2 ^ k) , {k, 1, **Length**[**CC**]]] - $\frac{30}{49} \text{Log}[2]^2$,
75]] , {t, 900, 950, 25}]]]

$$\text{Out[]:= } -0.9722271378$$

-0.972346879541577190222456920289273745968701261057647703257167189830252079216

-0.972346879541577190222456920289273745968701261057647703257167189830252079216

-0.972346879541577190222456920289273745968701261057647703257167189830252079216

$$\text{In[]:= } \mathbf{A222}^{(1,1,0)} [1, 0, 1] :=$$

$$-0.97234687954157719022245692028927374596870126105764770325716718983025207921637954823 \cdot 1505572162349^75 . ;$$

$$\mathbf{A222}^{(1,1,1)} [1, 0, 1] :=$$

$$2.669073090542368890429928158716709312982940521650435424346703583671623039891572526148001 \cdot 29547396^75 . ;$$

$$\mathbf{A222}^{(2,0,0)} [1, 0, 1] :=$$

$$3.477103517494925093640244486284714453212584966506884809535302065617610635029598419913002 \cdot 810379413^75 . ;$$

In[]:= Simplify [**D** [**D** [**Log** [**A22** [s, w1, w2]]] , s] , s] // . {s $\rightarrow 1$, w1 $\rightarrow 0$, w2 $\rightarrow 1$ }]

$$\text{Out[]:= } \frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-5 + 7 \text{Prime}[n] + 11 \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$

```

In[ ]:= N[Sum[
$$\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-5 + 7 \text{Prime}[n] + 11 \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
, {n, 1, 10^4}],
10]
Block[{$MaxExtraPrecision = 1250},
Do[CC = Join[{0}, Series[
$$\frac{\text{Prime}[n] (-5 + 7 \text{Prime}[n] + 11 \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
 //.,
Prime[n] -> 1/x, {x, 0, t}][[3]]];
Print[N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] +

$$\frac{\text{Log}[\text{Prime}[n]]^2 \text{Prime}[n] (-5 + 7 \text{Prime}[n] + 11 \text{Prime}[n]^2 + 2 \text{Prime}[n]^3)}{(1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3)^2}$$
 //.,
Prime[n] -> 2, 75]], {t, 900, 1000, 50}]]

```

Out[]:= 3.476864029

3.47710351749492509364024448628471445321258496650688480953530206561761063503

3.47710351749492509364024448628471445321258496650688480953530206561761063503

3.47710351749492509364024448628471445321258496650688480953530206561761063503

In[]:= **A222**^(2,0,0)[1, 0, 1] :=

3.477103517494925093640244486284714453212584966506884809535302065617610635029598419913;
002810379413`75.;

A222^(2,1,0)[1, 0, 1] :=

-2.9499695398586968685294197071571023563767599517216993445114516069274798788555450088943;
80831862888`75.;

Computing the 2nd residue

In[]:= **Simplify**[-3 Residue[Residue[Residue[f2[s, w1, w2], {s, 1}], {w2, 1}], {w1, 0}]]

$$\begin{aligned} \text{Out[]} = & -\frac{1}{2} X \left(A2[1, 0, 1] \text{Log}[X]^3 + \text{Log}[X]^2 \left((-4 + 9 \text{EulerGamma}) A2[1, 0, 1] + \right. \right. \\ & 2 A2^{(0,0,1)}[1, 0, 1] + 3 A2^{(0,1,0)}[1, 0, 1] + 2 A2^{(1,0,0)}[1, 0, 1] \left. \right) + \\ & \text{Log}[X] \left(6 A2[1, 0, 1] \left(1 - 4 \text{EulerGamma} + 4 \text{EulerGamma}^2 - \text{StieltjesGamma}[1] \right) + \right. \\ & 4 (-1 + 3 \text{EulerGamma}) A2^{(0,0,1)}[1, 0, 1] + A2^{(0,0,2)}[1, 0, 1] - 12 A2^{(0,1,0)}[1, 0, 1] + \\ & 18 \text{EulerGamma} A2^{(0,1,0)}[1, 0, 1] + 6 A2^{(0,1,1)}[1, 0, 1] - 4 A2^{(1,0,0)}[1, 0, 1] + 12 \text{EulerGamma} \\ & A2^{(1,0,0)}[1, 0, 1] + 2 A2^{(1,0,1)}[1, 0, 1] + 6 A2^{(1,1,0)}[1, 0, 1] + A2^{(2,0,0)}[1, 0, 1] \left. \right) + \\ & 3 \left(6 \text{EulerGamma}^3 A2[1, 0, 1] - 6 (-1 + \text{StieltjesGamma}[1]) A2^{(0,1,0)}[1, 0, 1] - \right. \\ & 4 A2^{(0,1,1)}[1, 0, 1] + A2^{(0,1,2)}[1, 0, 1] + \\ & 6 \text{EulerGamma}^2 \left(-2 A2[1, 0, 1] + A2^{(0,0,1)}[1, 0, 1] + A2^{(0,1,0)}[1, 0, 1] + A2^{(1,0,0)}[1, 0, 1] \right) - \\ & 4 A2^{(1,1,0)}[1, 0, 1] + 2 A2^{(1,1,1)}[1, 0, 1] + \\ & \text{EulerGamma} \left(-6 A2[1, 0, 1] (-1 + \text{StieltjesGamma}[1]) - 4 A2^{(0,0,1)}[1, 0, 1] + \right. \\ & A2^{(0,0,2)}[1, 0, 1] - 12 A2^{(0,1,0)}[1, 0, 1] + 6 A2^{(0,1,1)}[1, 0, 1] - 4 A2^{(1,0,0)}[1, 0, 1] + \\ & \left. \left. 2 A2^{(1,0,1)}[1, 0, 1] + 6 A2^{(1,1,0)}[1, 0, 1] + A2^{(2,0,0)}[1, 0, 1] \right) + A2^{(2,1,0)}[1, 0, 1] \right) \end{aligned}$$

In[255]:= **R2 :=**

N[Simplify[-3 Residue[Residue[Residue[f2[s, w1, w2], {s, 1}], {w2, 1}], {w1, 0}], 75];
R2

$$\begin{aligned} \text{Out[256]} = & X \left(-0.063266608926767601976889273178228608115673882704683409079402158103515197 - \right. \\ & 1.09330341220669665355627596705458590858192020923406445008228391655543992 \text{Log}[X] - \\ & 0.95628153527069893216943463295053465287189680169495828858563037127779537 \text{Log}[X]^2 - \\ & \left. 0.108889358309768189161503757059723406565398877500677968824138201761813246 \text{Log}[X]^3 \right) \end{aligned}$$

Constants from the last residue

In[]:= **Simplify**[A3[1, 1, 1]]

Out[]:= 0.21777871661953637832300751411944681313079775500135593764827640352362649111225262055792;
54438235637657

$$\mathbf{A333}^{(0,0,1)}[1, 1, 1] := \mathbf{A111}^{(0,1,0)}[1, 0, 0];$$

In[]:= **Simplify**[D[Log[A33[s, w1, w2]], w2] /. {s -> 1, w1 -> 1, w2 -> 1}]

$$\text{Out[]} = \frac{\text{Log}[\text{Prime}[n]] (-1 + 2 \text{Prime}[n])}{1 - 3 \text{Prime}[n] + \text{Prime}[n]^2 + \text{Prime}[n]^3}$$

In[]:= **A333**^(0,0,1)[1, 1, 1] := **A111**^(0,1,0)[1, 0, 0];

A333^(0,1,0) [1, 1, 1] := A333^(0,0,1) [1, 1, 1];

In[]:= Simplify[D[Log[A33[s, w1, w2]], w1] /. {s → 1, w1 → 1, w2 → 1}]

$$\text{Out[]:= } \frac{\text{Log[Prime[n]]} (-1 + 2 \text{Prime[n]})}{1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3}$$

In[]:= A333^(0,1,0) [1, 1, 1] := A333^(0,0,1) [1, 1, 1];

A333^(0,0,2) [1, 1, 1] := A111^(0,2,0) [1, 0, 0];

In[]:= Simplify[D[D[Log[A33[s, w1, w2]], w2], w2] /. {s → 1, w1 → 1, w2 → 1}]

$$\text{Out[]:= } - \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]} (1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + 2 \text{Prime[n]}^3)}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[]:= A333^(0,0,2) [1, 1, 1] := A111^(0,2,0) [1, 0, 0];

A333^(0,2,0) [1, 1, 1] := A111^(0,2,0) [1, 0, 0];

In[]:= Simplify[D[D[Log[A33[s, w1, w2]], w1], w1] /. {s → 1, w1 → 1, w2 → 1}]

$$\text{Out[]:= } - \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]} (1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + 2 \text{Prime[n]}^3)}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[]:= A333^(0,2,0) [1, 1, 1] := A111^(0,2,0) [1, 0, 0];

A333^(0,1,1) [1, 1, 1] :=

**0.259481271795966468394823473078732963027903993440970656938310391997636484657409776724613:
359110604`75.;**

In[]:= Simplify[D[D[Log[A33[s, w1, w2]], w1], w2] /. {s → 1, w1 → 1, w2 → 1}]

$$\text{Out[]:= } \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]}^2 (-1 + 2 \text{Prime[n]})}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[]:= N[Sum[$\frac{\text{Log[Prime[n]]}^2 \text{Prime[n]}^2 (-1 + 2 \text{Prime[n]})}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$, {n, 1, 10^5}], 10]

Block[{\$MaxExtraPrecision = 1000}, Do[CC = Join[{0, 0},

Series[$\frac{\text{Prime[n]}^2 (-1 + 2 \text{Prime[n]})}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$ /. Prime[n] → 1/x, {x, 0, t}][[3]]];

*Print[N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] +*

$\frac{\text{Log[Prime[n]]}^2 \text{Prime[n]}^2 (-1 + 2 \text{Prime[n]})}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$ /. n → 1, 75]], {t, 1000, 1100, 50}]]]

Out[]:= 0.2594812718

0. 259481271795966468394823473078732963027903993440970656938310391997636484657

0. 259481271795966468394823473078732963027903993440970656938310391997636484657

Out[⁶]= \$Aborted

A333^(0,1,1) [1, 1, 1] :=

**0. 259481271795966468394823473078732963027903993440970656938310391997636484657409776724:
613359110604`75.;**

A333^(1,1,0) [1, 1, 1] := A222^(1,1,0) [1, 0, 1];

In[⁶]= **Simplif y[D[D[Log[A33[s, w1, w2]], w1], s] /. {s → 1, w1 → 1, w2 → 1}]**

$$\text{Out[⁶]= } - \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]} (-1 + 2 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[⁶]= **A333^(1,1,0) [1, 1, 1] := A222^(1,1,0) [1, 0, 1];**

A333^(1,0,1) [1, 1, 1] := A333^(1,1,0) [1, 1, 1];

In[⁶]= **Simplif y[D[D[Log[A33[s, w1, w2]], w2], s] /. {s → 1, w1 → 1, w2 → 1}]**

$$\text{Out[⁶]= } - \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]} (-1 + 2 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[⁶]= **A333^(1,0,1) [1, 1, 1] := A333^(1,1,0) [1, 1, 1];**

A333^(1,0,0) [1, 1, 1] := A111^(0,1,0) [1, 0, 0];

In[⁶]= **Simplif y[D[Log[A33[s, w1, w2]], s] /. {s → 1, w1 → 1, w2 → 1}]**

$$\text{Out[⁶]= } \frac{\text{Log[Prime[n]]} (-1 + 2 \text{Prime[n]})}{1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3}$$

A333^(1,0,0) [1, 1, 1] := A111^(0,1,0) [1, 0, 0];

A333^(2,0,0) [1, 1, 1] := A222^(2,0,0) [1, 0, 1];

In[⁶]= **Simplif y[D[D[Log[A33[s, w1, w2]], s], s] /. {s → 1, w1 → 1, w2 → 1}]**

$$\text{Out[⁶]= } \frac{\text{Log[Prime[n]]}^2 \text{Prime[n]} (-5 + 7 \text{Prime[n]} + 11 \text{Prime[n]}^2 + 2 \text{Prime[n]}^3)}{(1 - 3 \text{Prime[n]} + \text{Prime[n]}^2 + \text{Prime[n]}^3)^2}$$

In[⁶]= **A333^(2,0,0) [1, 1, 1] := A222^(2,0,0) [1, 0, 1];**

Computing the 3rd residue

In[251]:= **R3 := N[Simplify[Residue[Residue[Residue[f3[s, w1, w2], {s, 1}], {w2, 1}], {w1, 1}], 75];**
R3

Out[252]= $X \left(0.134262687043915039723245950916112097651389855832294462765524522726957915 + \right.$
 $0.27455649581457199090794389843461174669660160946104562964329290831248609 \text{ Log}[X] +$
 $0.108889358309768189161503757059723406565398877500677968824138201761813246 \text{ Log}[X]^2 \left. \right)$