Delta-method

Summary

In[ ]:=
```
mprime[u_] :=
  Simplify[Residue[Residue[Zeta[s + 1]^3 Zeta[w + 1]^3 u^(s + w) A[s, w], {s, 0}], {w, 0}]];
m3[X_] := Integrate[mprime[u], u] //. u → X;
m3[X]
```

Out[ ]=
$$0.28723664774661941722166461781464595016603627439722224961891390744731 66\,X + 0.67786331083298038854157108306273365600322232270413534868810242515989728\,X\,\text{Log}[X] + 2.02119605787987777943324240784753809467091508369917789267040603543880549\,X\,\text{Log}[X]^2 + 0.71011392905364474755395892667350537295819711946375750493984571535973907 7\,X\,\text{Log}[X]^3 + 0.05444467915488409458075187852986170328269943875033898441206910088090662 28\,X\,\text{Log}[X]^4$$

The constants

In[ ]:=
```
A[0, 0] :=
  0.217778716619536378323007514119446813130797755001355937648276403523626491112252620557\
   92544382356376569183393577480 32`100.;
SS^(0,1)[0, 0] :=
  2.52906617358092992925958712930189459230009223994439976118899256270135780066286477485 3\
   625094702443`75.;
SS^(0,2)[0, 0] :=
  -2.79373963278994981211769042308953937015408419381694195210996243309601195345221795262\
    4829356255765`75.;
SS^(1,0)[0, 0] := SS^(0,1)[0, 0];
SS^(2,0)[0, 0] := SS^(0,2)[0, 0];
SS^(2,1)[0, 0] :=
  13.92494983824642902345888845122275722601808764998998303908490909644705166229158505249\
   9046880855312`75.;
SS^(1,1)[0, 0] :=
  -6.48922408680258078796953160319355949714389995731279703269639517857967764507358592096\
    0349813071894`75.;
SS^(1,2)[0, 0] := SS^(2,1)[0, 0];
SS^(2,2)[0, 0] :=
  -51.56161231785462256850318387377181628967444054263058784448231724591466806901085419 21\
   75426106119782`75.;
A^(0,1)[s_, w_] := A[s, w] * SS^(0,1)[s, w];
A^(0,2)[s_, w_] := A[s, w] SS^(0,1)[s, w]^2 + A[s, w] SS^(0,2)[s, w];
A^(1,0)[s_, w_] := A[s, w] * SS^(1,0)[s, w];
A^(2,0)[s_, w_] := A[s, w] SS^(1,0)[s, w]^2 + A[s, w] SS^(2,0)[s, w];
A^(2,1)[s_, w_] := A[s, w] SS^(0,1)[s, w] SS^(1,0)[s, w]^2 + 2 A[s, w] SS^(1,0)[s, w] SS^(1,1)[s, w] +
   A[s, w] SS^(0,1)[s, w] SS^(2,0)[s, w] + A[s, w] SS^(2,1)[s, w];
A^(1,1)[s_, w_] := A[s, w] SS^(0,1)[s, w] SS^(1,0)[s, w] + A[s, w] SS^(1,1)[s, w];
A^(1,2)[s_, w_] := A[s, w] SS^(0,1)[s, w]^2 SS^(1,0)[s, w] + A[s, w] SS^(0,2)[s, w] SS^(1,0)[s, w] +
   2 A[s, w] SS^(0,1)[s, w] SS^(1,1)[s, w] + A[s, w] SS^(1,2)[s, w];
A^(2,2)[s_, w_] := A[s, w] SS^(0,1)[s, w]^2 SS^(1,0)[s, w]^2 +
   A[s, w] SS^(0,2)[s, w] SS^(1,0)[s, w]^2 + 4 A[s, w] SS^(0,1)[s, w] SS^(1,0)[s, w] SS^(1,1)[s, w] +
   2 A[s, w] SS^(1,1)[s, w]^2 + 2 A[s, w] SS^(1,0)[s, w] SS^(1,2)[s, w] +
   A[s, w] SS^(0,1)[s, w]^2 SS^(2,0)[s, w] + A[s, w] SS^(0,2)[s, w] SS^(2,0)[s, w] +
   2 A[s, w] SS^(0,1)[s, w] SS^(2,1)[s, w] + A[s, w] SS^(2,2)[s, w];
A^(0,1)[0, 0] := A^(0,1)[s, w] //. {s → 0, w → 0};
A^(0,2)[0, 0] := A^(0,2)[s, w] //. {s → 0, w → 0};
A^(1,0)[0, 0] := A^(1,0)[s, w] //. {s → 0, w → 0};
A^(2,0)[0, 0] := A^(2,0)[s, w] //. {s → 0, w → 0};
A^(2,1)[0, 0] := A^(2,1)[s, w] //. {s → 0, w → 0};
A^(1,1)[0, 0] := A^(1,1)[s, w] //. {s → 0, w → 0};
A^(1,2)[0, 0] := A^(1,2)[s, w] //. {s → 0, w → 0};
A^(2,2)[0, 0] := A^(2,2)[s, w] //. {s → 0, w → 0};
```

Calculating the constants

Definitions

In[ ]:=
```
AA[s_, w_] := Log[1 - (1 - p (1 - p^(-1-s))^3/(-1 + p)) (1 - p (1 - p^(-1-w))^3/(-1 + p))];
```

A[0,0]

In[ ]:=
```
A[p_] := 1 - 4 / p^2 + 4 / p^3 - 1 / p^4;
N[Product[A[p] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 250},
 Do[CC = Join[{0}, Series[Log[A[p] //. p → 1 / x], {x, 0, t}][[3]]];
  Print[N[Exp[Sum[CC[[k]] * (PrimeZetaP[k] - 1 / 2^k), {k, 2, Length[CC]}]] * A[2], 100]],
  {t, 100, 1000, 100}]]
```

Out[ ]=
```
0.2177793815

0.21777871661918132921292757560788834387506648649613810079931346089460564100874076623711580270484132
49

0.21777871661953637832294206442009651997254770522207803199417770001779775466794394035434455194900310
112

0.21777871661953637832300751411943075625372410339213654917474314630527787056196945348585181272941444
39

0.21777871661953637832300751411944681313079332538476176000869171055893520847934856548700922524844513
79

0.21777871661953637832300751411944681313079775500135463442269121814648445537546239180709709046999931
74

0.21777871661953637832300751411944681313079775500135593764827600416628150830046101203728479556480982
44

0.21777871661953637832300751411944681313079775500135593764827640352362636524426930477993076300224557
18

0.21777871661953637832300751411944681313079775500135593764827640352362649111225258006218168940628137
03

0.21777871661953637832300751411944681313079775500135593764827640352362649111225262055792543058827702
19
```

$AA^{(0,1)}[0, 0]$

In[ ]:=
```
Simplify[AA^(0,1)[0, 0]]
```

Out[ ]=
$$\frac{3\,(-1+2\,p)\,\text{Log}[p]}{1-3\,p+p^2+p^3}$$

In[ ]:=
```
N[Sum[AA^(0,1)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[AA^(0,1)[0, 0] / Log[p] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[
   N[-Sum[CC[[k]] * (PrimeZetaP'[k] + Log[2] / 2^k), {k, 1, Length[CC]}] + AA^(0,1)[0, 0] //.
     p → 2, 100]], {t, 1000, 1100, 100}]]
```

Out[ ]=
```
2.529008972

2.52906617358092992925958712930189459230009223994439976118899256270135780066286086855713345730
58-01924

2.52906617358092992925958712930189459230009223994439976118899256270135780066286086855713345730
57-85344
```

In[ ]:=
```
N[Sum[AA^(0,1)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 20]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[AA^(0,1)[0, 0] / Log[p] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[
   N[-Sum[CC[[k]] * (PrimeZetaP'[k] + Log[2] / 2^k), {k, 1, Length[CC]}] + AA^(0,1)[0, 0] //.
     p → 2, 75]], {t, 1000, 1500, 100}]]
```

Out[ ]=
```
2.5290089721020069460

2.52906617358092992925958712930189459230009223994439976118899256270135780066

2.52906617358092992925958712930189459230009223994439976118899256270135780066
```

Out[ ]=
```
$Aborted
```

In[ ]:=
```
SS^(0,1)[0, 0] :=
 2.52906617358092992925958712930189459230009223994439976118899256270135780066286477485
  3625094702443`75.;
```

$AA^{(0,2)}[0, 0]$

In[ ]:=
```
Simplify[AA^(0,2)[0, 0]]
```

Out[ ]=
$$-\frac{3\,p\,(-1+2\,p)\,\left(-1-p+p^2\right)\,\text{Log}[p]^2}{\left(1-3\,p+p^2+p^3\right)^2}$$

```
In[ ]:= N[Sum[AA^(0,2)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 20]
        Block[{$MaxExtraPrecision = 1000},
         Do[CC = Join[{0}, Series[AA^(0,2)[0, 0] / Log[p]^2 //. p → 1 / x, {x, 0, t}][[3]]];
          Print[
           N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] + AA^(0,2)[0, 0] //.
             p → 2, 100]], {t, 950, 1000, 25}]]
```

```
Out[ ]= -2.7930211906145838489

-2.7937396327899498121176904230895393701540841938169419521099624330960119534461618364651975178 44-
  256652

-2.7937396327899498121176904230895393701540841938169419521099624330960119534461618364651968378 18-
  977211
```

```
Out[ ]= $Aborted
```

```
In[ ]:= N[Sum[AA^(0,2)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 20]
        Block[{$MaxExtraPrecision = 1000},
         Do[CC = Join[{0}, Series[AA^(0,2)[0, 0] / Log[p]^2 //. p → 1 / x, {x, 0, t}][[3]]];
          Print[
           N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] + AA^(0,2)[0, 0] //.
             p → 2, 75]], {t, 500, 1000, 100}]]
```

```
Out[ ]= -2.7930211906145838489

-2.7937396327899498121176904230895393701540841948341603712604674570311736 5731

-2.7937396327899498121176904230895393701540841938169419525587171231339909 1670

-2.7937396327899498121176904230895393701540841938169419521099624332884884 7568

-2.7937396327899498121176904230895393701540841938169419521099624330960119 5353

-2.7937396327899498121176904230895393701540841938169419521099624330960119 5345

-2.7937396327899498121176904230895393701540841938169419521099624330960119 5345
```

```
In[ ]:= SS^(0,2)[0, 0] :=
          -2.7937396327899498121176904230895393701540841938169419521099624330960119534522179526 2
            4829356255765`75.;
```

$AA^{(1,0)}[0, 0]$

```
In[ ]:= Simplify[AA^(1,0)[0, 0]]
```

$$\text{Out[ ]=}\quad \frac{3\,(-1+2\,p)\,\text{Log}[p]}{1-3\,p+p^2+p^3}$$

```
In[ ]:= SS^(1,0)[0, 0] := SS^(0,1)[0, 0]
```

In[•]:=
```
Simplify[AA^(2,0)[0, 0]]
```

Out[•]=
$$-\frac{3\,p\,(-1+2\,p)\,\left(-1-p+p^2\right)\,\text{Log}[p]^2}{\left(1-3\,p+p^2+p^3\right)^2}$$

In[•]:=
```
SS^(2,0)[0, 0] := SS^(0,2)[0, 0]
```

In[•]:=
```
Simplify[AA^(2,1)[0, 0]]
```

Out[•]=
$$\frac{9\,p^4\,\left(-3+5\,p-p^2+p^3\right)\,\text{Log}[p]^3}{\left(1-3\,p+p^2+p^3\right)^3}$$

```
N[Sum[AA^(2,1)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,1)[0, 0] / Log[p]^3] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[-Sum[CC[[k]] * (PrimeZetaP'''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,1)[0, 0] //. p → 2, 100]], {t, 950, 1000, 50}]]
```

In[•]:=
```
N[Sum[AA^(2,1)[0, 0] //. p → Prime[n], {n, 1, 10^4}], 20]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,1)[0, 0] / Log[p]^3] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[-Sum[CC[[k]] * (PrimeZetaP'''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,1)[0, 0] //. p → 2, 75]], {t, 500, 1000, 100}]]
```

Out[•]=
```
13.911328605724191288
```

```
13.924949838246429023458888451222757226018088069323583104094676792781002099 9
13.924949838246429023458888451222757226018087649989983261053155152331063762 8
13.924949838246429023458888451222757226018087649989983039084909207511316001 9
13.924949838246429023458888451222757226018087649989983039084909096447051715 6
13.924949838246429023458888451222757226018087649989983039084909096447051662 3
```

In[•]:=
```
Block[{$MaxExtraPrecision = 1500},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,1)[0, 0] / Log[p]^3] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[-Sum[CC[[k]] * (PrimeZetaP'''[k] + Log[2]^3 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,1)[0, 0] //. p → 2, 75]], {t, 1000, 1100, 50}]]
```

```
13.924949838246429023458888451222757226018087649989983039084909096447051662 3
13.924949838246429023458888451222757226018087649989983039084909096447051662 3
13.924949838246429023458888451222757226018087649989983039084909096447051662 3
```

In[•]:=
```
SS^(2,1)[0, 0] :=
  13.924949838246429023458888451222757226018087649989983039084909096447051662291585052499046880855312`75.;
```

$AA^{(1,1)}[0, 0]$

```
In[ ]:= Simplify[AA^(1,1)[0, 0]]
```

$$\text{Out[ ]}= -\frac{9\,p^4\,\text{Log}[p]^2}{\left(1-3\,p+p^2+p^3\right)^2}$$

```
N[Sum[Simplify[AA^(1,1)[0, 0]] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(1,1)[0, 0] / Log[p]^2] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[
   N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] + AA^(1,1)[0, 0] //.
     p → 2, 100]], {t, 950, 1000, 50}]]
```

```
In[ ]:= N[Sum[Simplify[AA^(1,1)[0, 0]] //. p → Prime[n], {n, 1, 10^4}], 20]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(1,1)[0, 0] / Log[p]^2] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[
   N[Sum[CC[[k]] * (PrimeZetaP''[k] - Log[2]^2 / 2^k), {k, 1, Length[CC]}] + AA^(1,1)[0, 0] //.
     p → 2, 75]], {t, 500, 1000, 100}]]
```

```
Out[ ]= -6.4881464161345619910

-6.4892240868025807879695316031935594971438999583304806675973488245437133994 1
-6.4892240868025807879695316031935594971438999573127970331453209085325222531 6
-6.4892240868025807879695316031935594971438999573127970326963951787722170513 5
-6.4892240868025807879695316031935594971438999573127970326963951785796776451 5
-6.4892240868025807879695316031935594971438999573127970326963951785796776450 7
-6.4892240868025807879695316031935594971438999573127970326963951785796776450 7
```

```
In[ ]:= SS^(1,1)[0, 0] :=
  -6.4892240868025807879695316031935594971438999573127970326963951785796776450735859209 6
    0349813071894`75.;
```

$AA^{(1,2)}[0, 0]$

```
In[ ]:= Simplify[AA^(1,2)[0, 0] - AA^(2,1)[0, 0]]
```

```
Out[ ]= 0
```

```
In[ ]:= SS^(1,2)[0, 0] := SS^(2,1)[0, 0]
```

$AA^{(0,1)}[0, 0]$

```
In[ ]:= Simplify[AA^(0,1)[0, 0] - AA^(1,0)[0, 0]]
Out[ ]= 0

In[ ]:= SS^(0,1)[0, 0] := SS^(1,0)[0, 0];
```

$AA^{(2,2)}[0, 0]$

```
In[ ]:= Simplify[AA^(2,2)[0, 0]]
```

$$\text{Out[ ]}= -\frac{9\,p^4\left(9-30\,p+31\,p^2-16\,p^3+29\,p^4-2\,p^5+p^6\right)\text{Log}[p]^4}{\left(1-3\,p+p^2+p^3\right)^4}$$

```
N[Sum[Simplify[AA^(2,2)[0, 0]] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,2)[0, 0] / Log[p]^4] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[Sum[CC[[k]] * (PrimeZetaP''''[k] - Log[2]^4 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,2)[0, 0] //. p → 2, 100]], {t, 950, 1000, 50}]]

In[ ]:= N[Sum[Simplify[AA^(2,2)[0, 0]] //. p → Prime[n], {n, 1, 10^4}], 20]
Block[{$MaxExtraPrecision = 1000},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,2)[0, 0] / Log[p]^4] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[Sum[CC[[k]] * (PrimeZetaP''''[k] - Log[2]^4 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,2)[0, 0] //. p → 2, 75]], {t, 500, 1000, 100}]]
Out[ ]= -51.388203427794243631

-51.561612317854622568503183873771816289674613285529907847017998095297244239 9

-51.561612317854622568503183873771816289674440542630697572801621837349662211 8

-51.561612317854622568503183873771816289674440542630587844482381300954372523 8

In[ ]:= Block[{$MaxExtraPrecision = 1500},
 Do[CC = Join[{0}, Series[Simplify[AA^(2,2)[0, 0] / Log[p]^4] //. p → 1 / x, {x, 0, t}][[3]]];
  Print[N[Sum[CC[[k]] * (PrimeZetaP''''[k] - Log[2]^4 / 2^k), {k, 1, Length[CC]}] +
      AA^(2,2)[0, 0] //. p → 2, 75]], {t, 1000, 1150, 50}]]

-51.561612317854622568503183873771816289674440542630587844482317245914668069 0

-51.561612317854622568503183873771816289674440542630587844482317245914668069 0
Out[ ]= $Aborted

In[ ]:= SS^(2,2)[0, 0] :=
  -51.561612317854622568503183873771816289674440542630587844482317245914668069010854192175426106119782`75.;
```

```
A[p_] := 1 - 4/p^2 + 4/p^3 - 1/p^4;
N[Product[A[p] //. p → Prime[n], {n, 1, 10^4}], 10]
Block[{$MaxExtraPrecision = 350},
 Do[CC = Join[{0}, Series[Log[A[p] //. p → 1 / x], {x, 0, t}][[3]];
  Print[N[Exp[Sum[CC[[k]] * (PrimeZetaP[k] - 1 / 2^k), {k, 2, Length[CC]}]] * A[2], 100]],
  {t, 1100, 1200, 50}]]
```

Out[•]= 0.2177793815

0.21777871661953637832300751411944681313079775500135593764827640352362649111225262055792544382357-637657

0.21777871661953637832300751411944681313079775500135593764827640352362649111225262055792544382357-637657

0.21777871661953637832300751411944681313079775500135593764827640352362649111225262055792544382357-637657