L-functions from an analytic perspective

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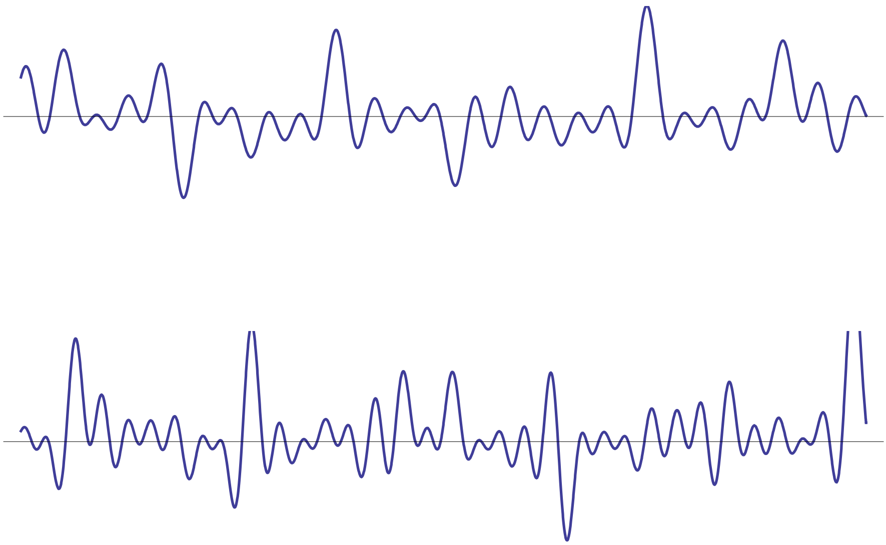
$Z(t)$ in two regions



Facts:

- $N(T) = \frac{1}{2\pi} T \log\left(\frac{T}{2\pi e}\right) + O(\log t)$,
  where $N(T) =$ number of zeros in $[0, T]$.
  So the average gap $\sim 2\pi / \log T$,
- $\frac{1}{T} \int_0^T Z(t)\, dt \;\to\; 0$ as $T \to 0$.

Conjectures:

- $Prob(\gamma_{j+1} - \gamma_j < \frac{\varepsilon}{\log j}) \approx \varepsilon^3$, where the zeros are $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$.
- $Z(t)$ gets as large as $\exp(\sqrt{\frac{1}{2} \log T \log\log T})$, and no larger.
- Local maxima of $Z(t)$ are positive, and local minima are negative.

The Hardy Z-function

The Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \sigma > 1 \tag{1}$$

Meromorphic continuation to all $s \in \mathbb{C}$.
Functional equation:

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s)$$

Equivalently,

$$\zeta(s) = \chi(s)\zeta(1-s)$$

Equivalently,

$$Z(t) := \chi(\tfrac{1}{2}+it)^{-1/2}\zeta(\tfrac{1}{2}+it)$$

is real if t is real.

The spacing between zeros

$N(T)$: the number of zeros in $0 < t < T$.

$$N(T) = \frac{1}{2\pi} T \log\left(\frac{T}{2\pi e}\right) + S(t)$$

$$S(T) = O(\log T)$$

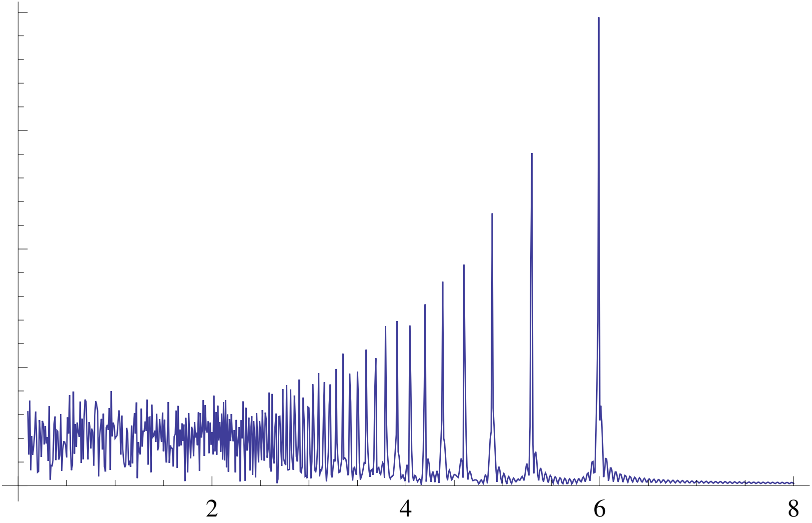
Average gap between zeros: solve $N(T+\delta) - N(T) \approx 1$.

$$\delta N'(T) \approx 1$$

$$\delta \approx \frac{1}{N'(T)} = \frac{2\pi}{\log(T/2\pi)}$$

Mathematica demo about rigidity in the zero spacing.

Fourier transform of the Z-function



Unsolved problems on zero spacing

Let $\tilde{\gamma} = \frac{1}{2\pi}\gamma \log(\gamma)$, so that
$\tilde{\gamma}_{j+1} - \tilde{\gamma}_j$ is 1 on average.

- Show that $\tilde{\gamma}_{j+1} - \tilde{\gamma}_j < 0.49$ a positive proportion of the time. (Related to $L(1,\chi)$; Thursday lecture.) Current record: 0.51???.
- Show that for all $A > 1$, there exists j so that

  $$\tilde{\gamma}_{j+1} - \tilde{\gamma}_j > A$$

  Current record: maybe 3?

Some methods involve comparing

$$\int_0^T M(t)^2 Z(t)^2 dt \quad \text{to} \quad \sum_{\gamma < T} \int_{-c/\log T}^{c/\log T} M(\gamma + t)^2 Z(\gamma + t)^2 dt$$