A Necessary and Sufficient Condition for Reality of Eigenvalues of Anharmonic Oscillators

Kwang C. Shin
University of West Georgia
AIM Workshop on Mathematical Aspects of Physics with Non-self-adjoint Operators
Half-Line Problem

Consider

$$\mathcal{H} := -\frac{d^2}{dx^2} + V(x)$$

in $L^2([0, \infty))$ with the Robin BCs at $x = 0$, that is,

$$y(0) \cos \theta + y'(0) \sin \theta = 0,$$

where $V(x) = x^m + a_1 x^{m-1} + \cdots + a_m$, $a_j \in \mathbb{C}$ and $\theta \in \mathbb{C}$.

• If $V$ and $\theta$ are real, then the eigenvalues are all real.

Is the converse true?
Theorem 1. If $\mathcal{H}$ is self-adjoint if and only if $\mathcal{H}$ has infinitely many real eigenvalues.

Corollary 2. If $\mathcal{H}$ has infinitely many real eigenvalues, then all eigenvalues are real.

**Question 1:** Is there any non-polynomial potentials $V(x)$ such that while $\mathcal{H}$ is non-self-adjoint, it has infinitely many real eigenvalues?

**Question 2:** Can we find a necessary and sufficient condition for reality of the eigenvalues of some classes of non-polynomial potentials?
Eigenvalue Problems in the Complex Plane

For an integer \( \ell \) in \( 1 \leq \ell \leq m - 1 \) fixed, consider

\[
(H_\ell u)(z) = \left[ -\frac{d^2}{dz^2} + (-1)^\ell (iz)^m - P(iz) \right] u(z) = \lambda u(z)
\]

with BCs \( u(z) \to 0 \) as \( z \to \infty \) in \( \mathbb{C} \) along the two rays

\[
\arg(z) = -\frac{\pi}{2} \pm \frac{(\ell + 1)\pi}{m + 2}, \quad \text{where}
\]

\[
P(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m, \quad a_j \in \mathbb{C}.
\]
Figure 1: Boundary conditions/Stokes sectors
\textbf{\(\mathcal{PT}\)-symmetric \(H_\ell\)}

- \(H_\ell\) is \(\mathcal{PT}\)-symmetric if and only if \(a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m\).
- \(\mathcal{PT}\)-symmetric \(H_\ell\) are not self-adjoint in general.
- If \(H_\ell\) is \(\mathcal{PT}\)-symmetric, then \(\lambda\) an eigenvalue of \(H_\ell\) \iff \overline{\lambda}\) an eigenvalue of \(H_\ell\).

Also,

- Eigenvalues are invariant under \(z \mapsto z - z_0\) for fixed \(z_0 \in \mathbb{C}\).
- When \(m\) even and \(\ell = \frac{m}{2}\), the BCs implies \(u \in L^2(\mathbb{R})\).
- We number \(\{\lambda_n\}_{n \geq 0}\) in the order of their magnitudes.
• \( \lim_{n \to \infty} \arg(\lambda_n) = 0. \)

• \( \lim_{n \to \infty} |\lambda_{n+1} - \lambda_n| = \infty. \)

• \( |\lambda_n| < |\lambda_{n+1}| \) for all large \( n. \)

• The geometric multiplicities of the eigenvalues are always 1.

• The algebraic multiplicities of the large eigenvalues are 1.
Theorem 3. If $H_\ell$ is $\mathcal{PT}$-symmetric, then the eigenvalues are all real with at most finitely many exceptions.

Theorem 4. Suppose that $\gcd(m, \ell) = 1$. Then the anharmonic oscillator $H_\ell$ with a potential $V(z)$ has infinitely many real eigenvalues if and only if $H_\ell$ with the potential $V(z - z_0)$ is $\mathcal{PT}$-symmetric for some $z_0 \in \mathbb{C}$.

• Suppose that $\gcd(m, \ell) = 1$. Then $H_\ell$ cannot have infinitely many real eigenvalues as well as infinitely many non-real eigenvalues. That is, if $H$ has infinitely many real eigenvalues, then $H$ can have at most finitely many non-real eigenvalues.
**Theorem 5.** Suppose that $m = 4$. Then $H$ has infinitely many real eigenvalues if and only if $H$ with $V(z - z_0)$ for some $z_0 \in \mathbb{C}$ is self-adjoint or $\mathcal{PT}$-symmetric.

Can we extend this to non-polynomial potentials?

**Question 3:** Is there any non-polynomial potentials $V(z)$ such that while $H$ is non-self-adjoint, it has infinitely many real eigenvalues?

**Question 4:** Can we find a necessary and sufficient condition for reality of the eigenvalues of some classes of non-polynomial potentials?
Thank you!