A Necessary and Sufficient Condition for Reality of Eigenvalues of Anharmonic Oscillators

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AIM Workshop on Mathematical Aspects of Physics with Non-self-adjoint Operators

Half-Line Problem

Consider

$$\mathcal{H} := -\frac{d^2}{dx^2} + V(x)$$

in $L^2([0,\infty))$ with the Robin BCs at $x = 0$, that is,

$$y(0)\cos\theta + y'(0)\sin\theta = 0,$$

where $V(x) = x^m + a_1 x^{m-1} + \cdots + a_m$, $a_j \in \mathbb{C}$ and $\theta \in \mathbb{C}$.

- If V and θ are real, then the eigenvalues are all real.

  Is the converse true?

Theorem 1. *If $\mathcal{H}$ is self-adjoint if and only if $\mathcal{H}$ has infinitely many real eigenvalues.*
Corollary 2. *If $\mathcal{H}$ has infinitely many real eigenvalues, then all eigenvalues are real.*

Question 1: Is there any non-polynomial potentials $V(x)$ such that while $\mathcal{H}$ is non-self-adjoint, it has infinitely many real eigenvalues?

Question 2: Can we find a necessary and sufficient condition for reality of the eigenvalues of some classes of non-polynomial potentials?

Eigenvalue Problems in the Complex Plane

For an integer ℓ in $1 \leq \ell \leq m-1$ fixed, consider

$$(H_\ell u)(z) = \left[-\frac{d^2}{dz^2} + (-1)^\ell (iz)^m - P(iz) \right] u(z) = \lambda u(z)$$

with BCs $u(z) \to 0$ as $z \to \infty$ in $\mathbb{C}$ along the two rays

$$\arg(z) = -\frac{\pi}{2} \pm \frac{(\ell+1)\pi}{m+2}, \quad \text{where}$$

$$P(z) = a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_{m-1} z + a_m, \quad a_j \in \mathbb{C}.$$

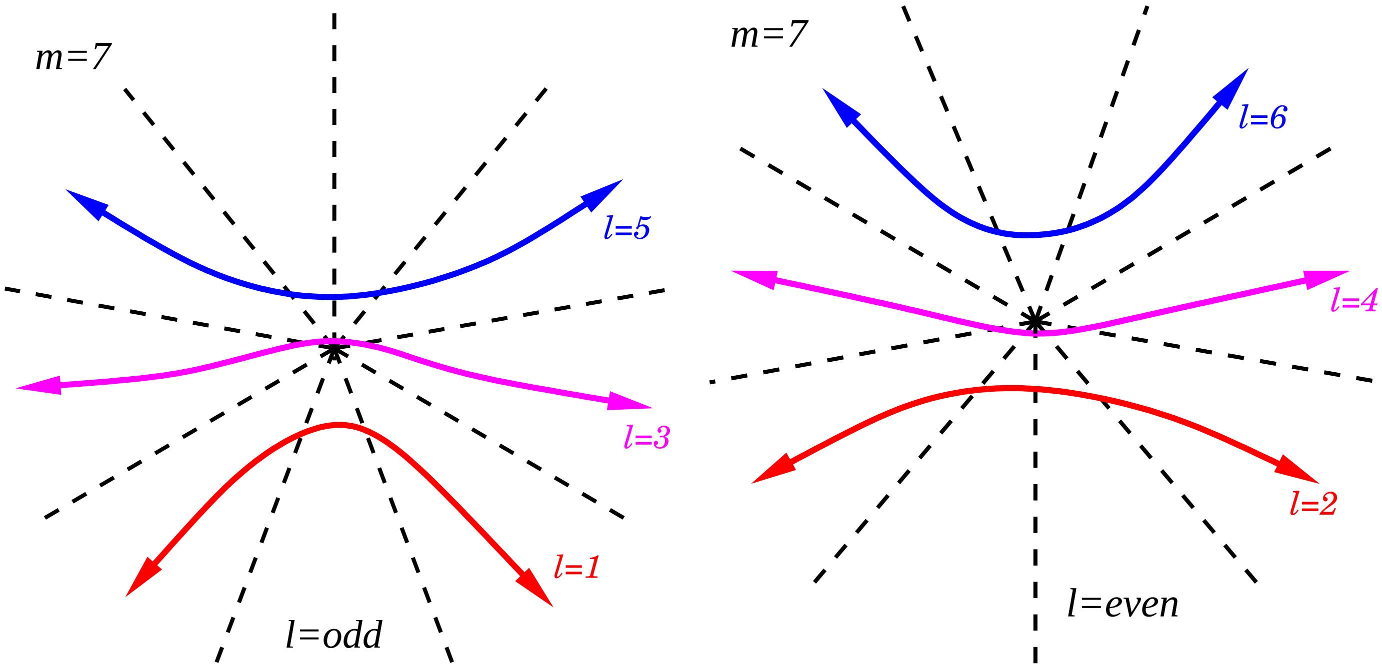

Figure 1: Boundary conditions/Stokes sectors

$\mathcal{PT}$-symmetric H_ℓ

- H_ℓ is $\mathcal{PT}$-symmetric if and only if $a = (a_1, a_2, \ldots, a_m) \in \mathbb{R}^m$.
- $\mathcal{PT}$-symmetric H_ℓ are not self-adjoint in general.
- If H_ℓ is $\mathcal{PT}$-symmetric, then λ an eigenvalue of $H_\ell \Longleftrightarrow \overline{\lambda}$ an eigenvalue of H_ℓ.

Also,

- Eigenvalues are invariant under $z \mapsto z - z_0$ for fixed $z_0 \in \mathbb{C}$.
- When m even and $\ell = \frac{m}{2}$, the BCs implies $u \in L^2(\mathbb{R})$.
- We number $\{\lambda_n\}_{n \geq 0}$ in the order of their magnitudes.

- $\lim_{n\to\infty} \arg(\lambda_n) = 0$.
- $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = \infty$.
- $|\lambda_n| < |\lambda_{n+1}|$ for all large n.
- The geometric multiplicities of the eigenvalues are always 1.
- The algebraic multiplicities of the large eigenvalues are 1.

Theorem 3. *If H_ℓ is $\mathcal{PT}$-symmetric, then the eigenvalues are all real with at most finitely many exceptions.*

Theorem 4. *Suppose that* $\gcd(m,\ell)=1$. *Then the anharmonic oscillator H_ℓ with a potential $V(z)$ has infinitely many real eigenvalues if and only if H_ℓ with the potential $V(z-z_0)$ is $\mathcal{PT}$-symmetric for some $z_0\in\mathbb{C}$.*

• Suppose that $\gcd(m,\ell)=1$. Then H_ℓ cannot have infinitely many real eigenvalues as well as infinitely many non-real eigenvalues. That is, if H has infinitely many real eigenvalues, then H can have at most finitely many non-real eigenvalues.

Theorem 5. *Suppose that $m = 4$. Then H has infinitely many real eigenvalues if and only if H with $V(z-z_0)$ for some $z_0 \in \mathbb{C}$ is self-adjoint or $\mathcal{PT}$-symmetric.*

Can we extend this to non-polynomial potentials?

Question 3: Is there any non-polynomial potentials $V(z)$ such that while H is non-self-adjoint, it has infinitely many real eigenvalues?

Question 4: Can we find a necessary and sufficient condition for reality of the eigenvalues of some classes of non-polynomial potentials?

Thank you!