Lieb-Thirring type estimates for non-selfadjoint operators

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Contents

Preliminaries

$\mathcal{H}$ complex Hilbert space, Z closed linear operator in $\mathcal{H}$ with **spectrum** $\sigma(Z)$ and **resolvent set** $\varrho(Z) = \mathbb{C} \setminus \sigma(Z)$.

- $\sigma_d(Z) := \{\lambda \in \mathbb{C} : \lambda \text{ isolated eigenvalue of finite algebraic mult.}\}$.
- $\sigma_{ess}(Z) := \{\lambda \in \mathbb{C} : \lambda I - Z \text{ not Fredholm}\}$.

Fact 1: If $K \in \mathcal{K}(\mathcal{H})$, then $\sigma_{ess}(Z + K) = \sigma_{ess}(Z)$.

Fact 2: $\sigma_{ess}(Z) \cap \sigma_d(Z) = \emptyset$.

Fact 3: If $\varrho(Z) \neq \emptyset$ and $\sigma_{ess}(Z) \subset [a, \infty), a \in \mathbb{R}$, then

$$\sigma(Z) = \sigma_d(Z) \,\dot{\cup}\, \sigma_{ess}(Z)$$

and discrete eigenvalues can accumulate at $\sigma_{ess}(Z)$ only.

The s.a. Lieb-Thirring inequalities

- $H_0 := -\Delta$ in $L^2(\mathbb{R}^d)$, with $\mathrm{Dom}(H_0) = W^{2,2}(\mathbb{R}^d)$.
  Then $H_0 = H_0^*$, $H_0 \geq 0$ and $\sigma(H_0) = \sigma_{ess}(H_0) = [0, \infty)$.

- $H := H_0 + V$, where

$$V \in L^p(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d, \mathbb{R}) \quad \text{with} \quad \begin{cases} p \geq d/2, & d \geq 3 \\ p > 1, & d = 2 \\ p \geq 1, & d = 1. \end{cases}$$

  Then $H = H^*$ and $H \geq -c_H$ with $c_H \geq 0$.

- Moreover: $\sigma_{ess}(H) = \sigma_{ess}(H_0) = [0, \infty)$ and

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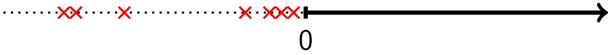

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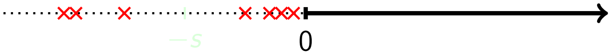

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With p as above the following holds:

$$p \neq d/2 : \qquad \sum_{\lambda \in \sigma_d(H)} |\lambda|^{p-d/2} \leq C_{p,d} \|V_-\|^p_{L^p},$$

$$p = d/2 : \qquad N_H(0) \leq C_d \|V_-\|^{d/2}_{L^{d/2}} \qquad \text{(CLR-bound)}$$

- Used in proof of stability of matter,
- semi-classical interpretation,
- for the number of eigenvalues less than $-s$, $s > 0$, we obtain

$$N_H(-s) \leq \frac{1}{s^{p-d/2}} C_{p,d} \|V_-\|^p_{L^p}, \qquad s > 0.$$

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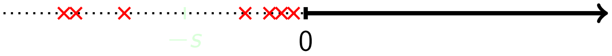

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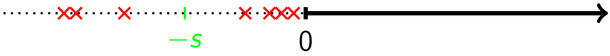

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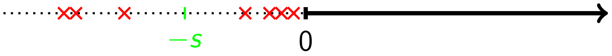

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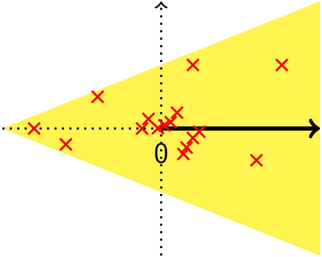
- Now we consider

$$V : \mathbb{R}^d \to \mathbb{C}, \quad \mathrm{Im}(V) \neq 0,$$

so $H = H_0 + V$ is **non-selfadjoint**.

- We still have $\sigma(H) = \sigma_d(H) \,\dot{\cup}\, [0, \infty)$, but the picture is different...



- **Numerical range** $\mathrm{Num}(H) = \{\langle Hf, f\rangle : f \in \mathrm{Dom}(H), \|f\| = 1\}$ is contained in a **sector** and $\sigma(H) \subset \overline{\mathrm{Num}(H)}$.
(For more precise bounds, see talk by Rupert Frank.)

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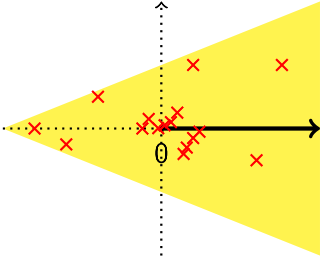

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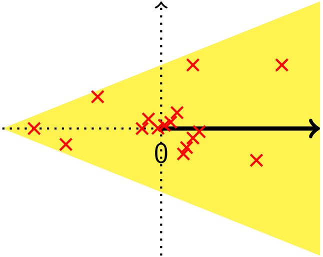

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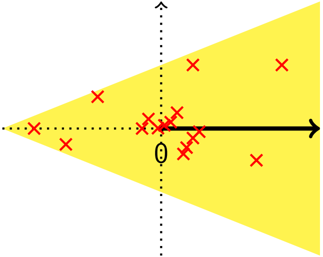

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Better:

$$\sum \frac{\mathrm{dist}(\lambda, [0,\infty))^p}{|\lambda|^{d/2}} \leq C_{p,d} \|V\|_{L^p}^p$$

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Conjecture: $$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^p}{|\lambda|^{d/2}} \leq C_{p,d} \|V\|_{L^p}^p$$

- Eigenvalues behave differently when approaching 0 (from left or right) or a point in $(0,\infty)$, respectively. For instance, for $\lambda_n \to \lambda_0 \in (0,\infty)$ we obtain
$$\sum_n |\operatorname{Im}(\lambda_n)|^p < \infty.$$
- If $\Omega_s := \{\lambda : \operatorname{dist}(\lambda, [0,\infty))^p \geq s|\lambda|^{d/2}\}$, then

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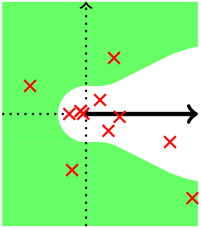

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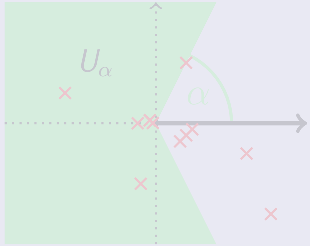
The n.s.a. Lieb-Thirring inequalities: some known results

Frank, Laptev, Lieb, Seiringer [FLLS06], $(p \geq d/2+1)$

For $\alpha \in (0, \pi/2)$:

$$\sum_{\lambda \in \sigma_d(H) \cap U_\alpha} |\lambda|^{p-d/2} \leq C(\alpha) C_{p,d} \|V\|_{L^p}^p,$$

where $C(\alpha) = \left(1 + \frac{2}{\tan(\alpha)}\right)^p$.



Multiplying left- and right-hand side with suitable weight $w(\alpha)$ and integrating over $(0, \pi/2)$:

Demuth, H., Katriel [DHK09], $(p \geq d/2+1$ and $\varepsilon > 0)$

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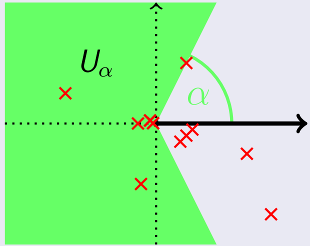

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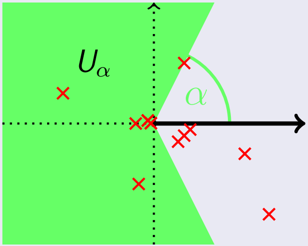

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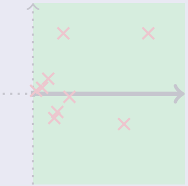

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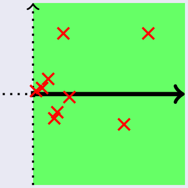

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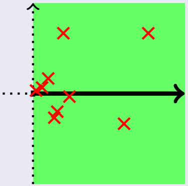

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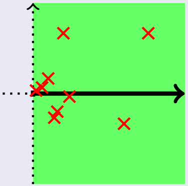

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- Further results by Laptev, Safronov [LS09], Safronov [Saf10].
- For 'small' values of p: Improvements by Frank, Sabin [FS14] (see next talk !?).

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Conjecture: $$\sum_{\lambda \in \sigma_d(H)} \frac{\mathrm{dist}(\lambda, [0,\infty))^p}{|\lambda|^{d/2}} \leq C_{p,d} \|V\|_{L^p}^p$$

Related questions/problems:

1. Is it true? For what p?
2. Independent of the conjecture: Given $V \in L^p$, what is the minimal q such that,e.g., for a sequence $\lambda_n \to \lambda_0 \neq 0$ we have
$$\sum_n |\mathrm{Im}(\lambda_n)|^q < \infty. \qquad \text{(see next talk !?)}$$
3. Can one say something for $p = d/2$? (see next talk !?)
Are there only finitely many eigenvalues in the left half-plane?
$$\sum_{\lambda \in \sigma_d(H), \mathrm{Re}(\lambda)<0} \frac{\mathrm{dist}(\lambda, [0,\infty))^{d/2}}{|\lambda|^{d/2}} = \sum_{\lambda \in \sigma_d(H), \mathrm{Re}(\lambda)<0} 1 = N_H(\mathbb{C}_-).$$
4. Construct examples!

Methods of proof

Essentially two (independent) methods of proof:

1. Use relations between eigen- and singular values of compact operators, in the spirit of Weyl's inequality
$$\sum_n |\lambda_n(K)|^p \leq \sum_n s_n(K)^p, \quad p > 0.$$
2. Relate eigenvalues to zeros of holomorphic function and use tools from complex analysis.

Both methods can be applied in more general situations.

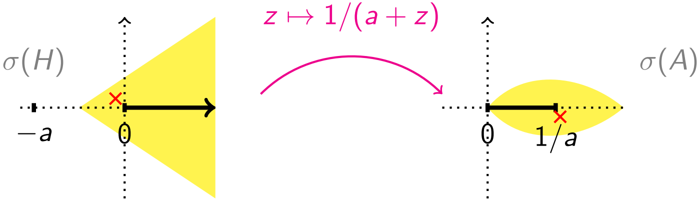
In the following: We try to give an idea of how methods work by proving some of the above results . . . starting with Method 1.

Reduction to bounded operators

Let $a > 0$ such that $-a \in \varrho(H_0) \cap \varrho(H)$ and define

$$A_0 := (a + H_0)^{-1}, \qquad A := (a + H)^{-1}.$$

- **Spectral mapping:** $\sigma(A_0) = \{(a+\lambda)^{-1} : \lambda \in \sigma(H_0)\}$, and the same is true for σ_d and σ_{ess} (and for A).



- **Hence:** $\sigma(A_0) = [0, 1/a] = \sigma_{ess}(A_0) = \sigma_{ess}(A)$.

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- **Note:** We will see in a minute, that the operator

$$A_0 - A \;=\; (a + H_0)^{-1} - (a + H)^{-1}$$

is **compact** (so $\sigma_{ess}(A) = \sigma_{ess}(A_0)$ and $\sigma_{ess}(H) = \sigma_{ess}(H_0)$).

- **Hence, our problem is of the following type:**

Given $B_0 \in \mathcal{B}(\mathcal{H})$ and $K \in \mathcal{K}(\mathcal{H})$, study rate of approximation of discrete eigenvalues of

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Degrees of compactness

Singular/approximation numbers of $K \in \mathcal{B}(\mathcal{H})$:

$$s_n(K) := \inf\{\|K - F\| : \text{Rank}(F) < n\}, \quad n \in \mathbb{N}.$$

- $\|K\| = s_1(K) \geq s_2(K) \geq \dots$
- $K \in \mathcal{K}(\mathcal{H})$ iff $s_n(K) \to 0$.

von Neumann-Schatten classes: For $0 < q < \infty$

$$S_q(\mathcal{H}) := \{K \in \mathcal{B}(\mathcal{H});\ \|K\|_q := \|(s_n(K))\|_{l^q} < \infty\}$$

- If $q_1 < q_2$, then $S_{q_1}(\mathcal{H}) \subset S_{q_2}(\mathcal{H}) \subset \mathcal{K}(\mathcal{H})$.
- $S_q(\mathcal{H})$ is **ideal** in $\mathcal{B}(\mathcal{H})$ and $\|RKT\|_q \leq \|R\|\|K\|_q\|T\|$ if $R, T \in \mathcal{B}(\mathcal{H})$.

More specific problem: Given $B_0 \in \mathcal{B}(\mathcal{H})$ and $K \in S_q(\mathcal{H})$, study rate of approximation of discrete eigenvalues of $B := B_0 + K$ to $\sigma_{ess}(B_0)$.

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S_q-properties of $A_0 - A$

$$\begin{aligned} A_0 - A &= (a + H_0)^{-1} - (a + H)^{-1} \\ &= (a + H)^{-1} V (a + H_0)^{-1} \end{aligned}$$

- Let $g_a(t) := (a + |t|^2)^{-1}, t \in \mathbb{R}^d$. Then

$$(a + H_0)^{-1} = \mathcal{F}^{-1} M_{g_a} \mathcal{F},$$

where $\mathcal{F} : L^2(\mathbb{R}^d, dx) \to L^2(\mathbb{R}^d, dt)$ denotes the Fourier transform.

- Classical estimate: (Seiler,Simon)

$$\|V(a + H_0)^{-1}\|_q = \|V\mathcal{F}^{-1} M_{g_a} \mathcal{F}\|_q \leq C_{q,d} \|V\|_{L^q} \|g_a\|_{L^q},$$

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Theorem ([Han11]): If $K \in S_q(\mathcal{H})$, $q \geq 1$, then

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- $\operatorname{Num}(B_0) = \{\langle B_0 f, f\rangle : f \in \mathcal{H}, \|f\| = 1\}$, convex set.
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An estimate involving the numerical range

$$\sum_{\lambda \in \sigma_d(B)} \operatorname{dist}(\lambda, \operatorname{Num}(B_0))^q \leq \|K\|_q^q, \qquad B = B_0 + K, \quad q \geq 1.$$

Proof:

- **Lemma:** For every ONS $(f_n) \subset \mathcal{H}$ we have $\sum_n |\langle K f_n, f_n \rangle|^q \leq \|K\|_q^q$.
- **Schur's lemma**: Let $|\lambda_1| \geq |\lambda_2| \geq \ldots$ denote the discrete eigenvalues of B. Then there exists an orthonormal sequence $\{e_n\} \subset \mathcal{H}$ and a sequence $\{b_{jk}\} \subset \mathbb{C}$, such that

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- **Consequently:**

$$\begin{aligned} \|K\|_q^q &\geq \sum_n |\langle K e_n, e_n \rangle|^q = \sum_n |\langle (B - B_0) e_n, e_n \rangle|^q \\ &= \sum_n |\lambda_n - \langle B_0 e_n, e_n \rangle|^q \geq \sum_n \operatorname{dist}(\lambda_n, \operatorname{Num}(B_0))^q. \end{aligned}$$

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Let $H_0 = -\Delta$ and $H = H_0 + V$, where $\mathrm{Re}(V) \geq 0$ and

$$V \in L^p(\mathbb{R}^d, \mathbb{C}) \cap L^\infty(\mathbb{R}^d, \mathbb{C}) \quad \text{with} \quad \begin{cases} p > d/2, & d \geq 3 \\ p > 1, & d = 2 \\ p \geq 1, & d = 1. \end{cases}$$

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Remarks

$$\sum_{\lambda \in \sigma_d(B)} \operatorname{dist}(\lambda, \operatorname{Num}(B_0))^q \leq \|K\|_q^q, \quad q \geq 1.$$

- For suitable choices of B_0, B one can recover the results of [FLLS06].
- If B_0 is selfadjoint, $K \in S_q(\mathcal{H})$ with $q > 1$:

$$\sum_{\lambda \in \sigma_d(B)} \operatorname{dist}(\lambda, \sigma(B_0))^q \leq C_q \|K\|_q^q \qquad \text{(see [Han13])}.$$

- Both results have been applied to different H_0 (perturbed by a complex potential):
  - [Han11]: Jacobi operators,
  - Dubuisson [Dub14] : fractional Schrödinger and Dirac operators,
  - Sambou [Sam14]: magnetic Schrödinger operators,
  - Golinskii, Kupin [GK15]: periodic Schrödinger operators.

Method 2

Relate eigenvalues to zeros of holomorphic function and use tools from complex analysis.

Regularized Determinants

Let $K \in S_N(\mathcal{H})$, $N \in \mathbb{N}$, and define

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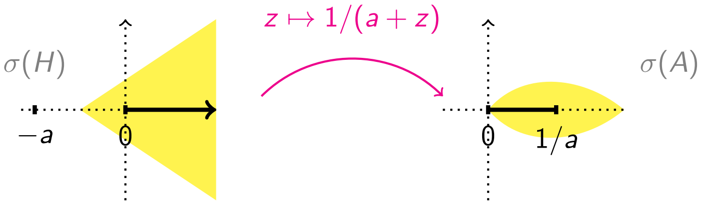
Now as above let $A_0 = (a+H_0)^{-1}, A = (a+H)^{-1}$ with $A - A_0 \in S_p(\mathcal{H})$.



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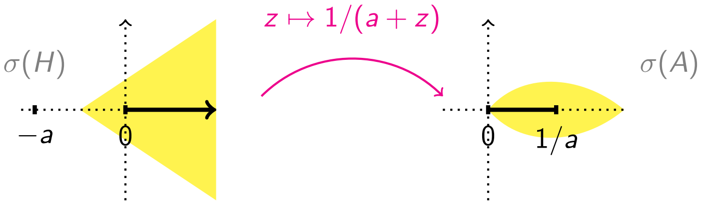

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- $h(-a) = \det_{\lceil p \rceil}(I) = 1.$
- $h(z) = 0$ iff $z \in \sigma_d(H)$ (and order and multiplicity coincide)
- **Upper bound:**

$$|h(z)| \leq \exp\left(\gamma_p \|(A - A_0)[(a+z)^{-1} - A_0]^{-1}\|_p^p\right).$$

Caution: We have p and not $\lceil p \rceil$!

Note that

$$(A - A_0)[(a+z)^{-1} - A_0]^{-1} = (a+z)(a+H)^{-1} V [z - H_0]^{-1}$$

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Intermezzo: Zeros of holomorphic functions on $\mathbb{D}$

Let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and $f \in \mathbb{H}(\mathbb{D})$ with $|f(0)| = 1$.

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Let $f \in \mathbb{H}(\mathbb{D})$, satisfying $|f(0)| = 1$ and (1). Then for every $\varepsilon > 0$

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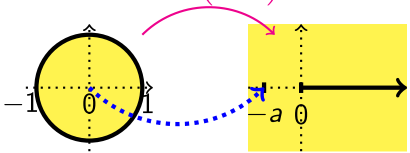

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An estimate obtained by the second method

to obtain...

Theorem (Demuth, H., Katriel [DHK09], $(p > d/2, d \geq 4)$)

Let $\varepsilon > 0$ *and let* $a > 0$ *with* $-a \in \varrho(H) \cap \varrho(H_0)$. *Set*

$$\begin{aligned} \eta_1 &= p + \varepsilon, \\ \eta_2 &= [(p - d + 1)_+ - 1 + \varepsilon]_+ \\ \eta_3 &= [(-p + 1 + d)_+ - 1 + \varepsilon]_+. \end{aligned}$$

Then

$$\sum_{\lambda \in \sigma_d(H)} \frac{\operatorname{dist}(\lambda, [0,\infty))^{p+\varepsilon}}{|\lambda|^{\frac{\eta_1 - \eta_2}{2}} (|\lambda| + a)^{\eta_1 + \frac{\eta_2 + \eta_3}{2}}} \leq C_{p,d}(\varepsilon) a^{d/2} \|(a+H)^{-1}\|^p \|V\|_{L^p}^p.$$

Now estimate resolvent from above and integrate with respect to a.

Remarks

- Borichev-Golinskii-Kupin-Theorem has been generalized to finitely-connected (Golinskii, Kupin [GK12]) and more general domains (Favorov,Golinskii [FG15]).
- Applications to different H_0:
  - H., Katriel [HK11]: Jacobi operators,
  - Dubuisson [Dub14]: fractional Schrödinger and Dirac operators,
  - Sambou [Sam14]: magnetic Schrödinger operators.
- Improvements of our results by Frank and Sabin [FS14] (see next talk).

Advertisements

- We have written a review article, comparing the results of the two methods:

Eigenvalues of non-selfadjoint operators: a comparison of two approaches (M. Demuth, M. Hansmann, G. Katriel), Oper. Theory Adv. Appl. (232), 107–163. Birkhäuser/Springer Basel AG, Basel, 2013.

- Currently we are working hard to transfer Method 2 to operators on Banach spaces. For a first result see our paper

Estimating the number of eigenvalues of linear operators on Banach spaces (M. Demuth, F. Hanauska, M. Hansmann, G. Katriel), J. Funct. Anal. 268 (2015), no.4, 1032-1052.

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