Spectral problems for linear pencils

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Based on joint works with
E Brian Davies (King's College London)
Linear Alg. Appl. 2014
and
Marcello Seri (University College London)
arXiv:1503.08615
(and maybe a bit more)

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Indefinite pencils

We consider a generalised spectral problem for a pair of self-adjoint operators A, B acting in a Hilbert space $\mathcal{H}$ by defining the operator *pencil*

$$\mathcal{P} = \mathcal{P}(\lambda) := A - \lambda B$$

depending on the spectral parameter λ; as usual, the *spectrum* of $\mathcal{P}$ is defined as the set of values λ for which there is no bounded inverse $\mathcal{P}(\lambda)^{-1}$ (in case of unbounded operators we assume for simplicity that the domain of $\mathcal{P}(\lambda)$ is independent of λ). λ is an *eigenvalue* of $\mathcal{P}$ if zero is an eigenvalue of $\mathcal{P}(\lambda)$. If B is invertible, then the spectrum of $\mathcal{P}$ coincides with the spectrum of (generally speaking, non-self-adjoint) operator $H = B^{-1}A$ and may, therefore, be non-real.

If, however, either A or B is sign-definite, then the problem may be be reduced to the one for a self-adjoint operator $\sqrt{B^{-1}}A\sqrt{B^{-1}}$, and the spectrum is real. Such a situation is very common in numerical analysis.

Indefinite pencils

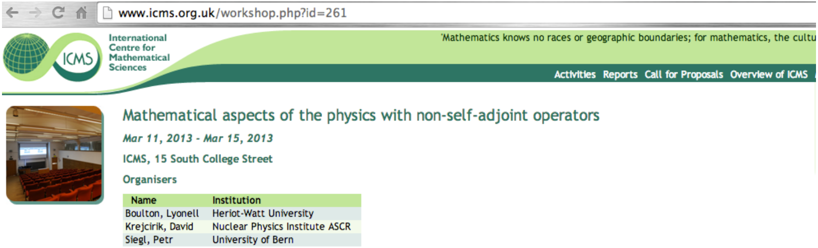
The situation is completely reversed when both operators A, B are sign-indefinite. In this case, the complex eigenvalues may (and often do) appear; the structure of, and the interaction between, the real and the non-real part of the spectrum, as one varies the parameters of the problem, can be extremely convoluted and rarely could be satisfactory explained by "soft" analysis. We usually need proper complex analysis plus other tricks!

I will proceed by discussing two distinct examples, a finite-dimensional one and an infinite-dimensional one. A third one, most physically relevant (a Dirac pencil describing graphene waveguides, see D. Elton, M.L., I. Polterovich, Ann. Henri Poincare (2014)) I will not have time for.

Acknowledgements

I begin with a continuous problem, which is in fact slightly easier.

- ICMS Workshop in Edinburgh in 2013



- In the Open Problems Session, the talk by Jussi Behrndt (see also Integral Equations Operator Theory **77** (2013), no. 3, 299–301) on the following problem:

Statement of the problem

Let

$$A = A^* := -\frac{d^2}{dx^2} - V(x)\cdot$$

$$\mathrm{Dom}(A) := \{u \in L^2(\mathbb{R}), u, u' \text{ abs. continuous}, Au \in L^2(\mathbb{R})\}$$

denote a one-dimensional Schrödinger operator in $L^2(\mathbb{R})$ with

1. $V \in L^1_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$;
2. $V(x) > 0, \quad x \in \mathbb{R}$;
3. $\mathrm{Spec}_{\mathrm{ess}}(A) = [0, +\infty)$;
4. $\mathrm{Spec}(A) \cap (-\infty, 0)$ consists of eigenvalues which accumulate to 0.

I believe that (1), (2) and

$$V(x) \sim |x|^{-\alpha}, \quad 1/2 < \alpha < 2$$

imply that both $\pm\infty$ are limit points, and therefore (3). I also assume for simplicity $V(x) = V(-x)$. I'll call A the definite Schrödinger operator.

Indefinite Schrödinger operator

Now let $B = \mathrm{sign}(x)\cdot$ be a multiplication operator by ± 1 on $\mathbb{R}_\pm$, resp., and consider the spectrum of

$$H := B^{-1}A, \qquad \mathrm{Dom}(H) = \mathrm{Dom}(A).$$

Thus, we consider the spectral problem

$$\boxed{Au := \left(-\frac{d^2}{dx^2} - V(x)\right) u(x) = \lambda\, \mathrm{sign}(x)\, u(x) := \lambda\, B u}$$

H is non-self-adjoint, but as $B = B^* = B^{-1}$, H can be viewed as a self-adjoint operator in a Kreĭn space with indefinite inner product

$$[u, v] := (Bu, v)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \mathrm{sign}(x) u \overline{v}\, \mathrm{d}x.$$

In a series of papers between 2007 and 2010, Behrndt, Katatbeh, Trunk and Karabash used this fact to prove a number of statements about operator A and its generalisations.

Indefinite Schrödinger operator (contd.)

In particular,

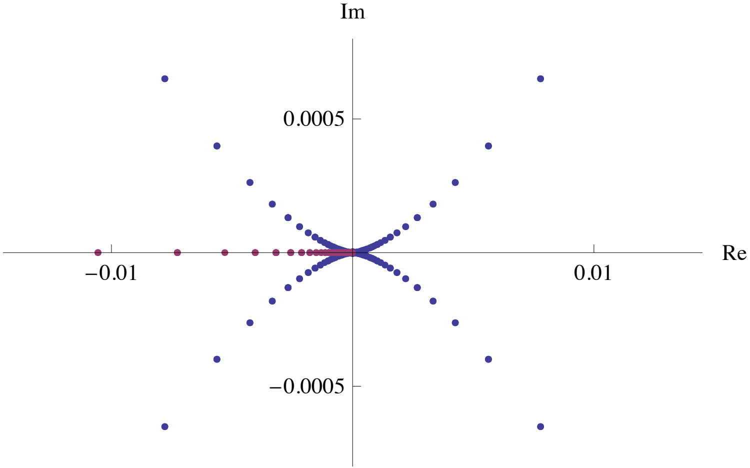
- $\mathrm{Spec}_{\mathrm{ess}}(H) = \mathbb{R}$;
- $\mathrm{Spec}(H) \setminus \mathbb{R}$ is symmetric w.r.t. $\mathbb{R}$ (and w.r.t. $\mathrm{i}\mathbb{R}$ if V is even) and consists of eigenvalues with only possible accumulation point at 0.
- there exist potentials for which 0 is indeed an accumulation point of non-real eigenvalues (non-constructive argument).

The still open problem is to describe the behaviour of complex eigenvalues of H near zero for a given potential. More motivation comes from a a numerical example of Behrndt, Katatbeh, Trunk (2008) for the particular potential

$$V_0(x) = \frac{1}{1 + |x|}$$

which I'll show on the next slide. The study of this potential is the main (and only) topic of the first part of my talk.

Numerics for $V_0(x) = \frac{1}{1+|x|}$



Sharp self-adjoint asymptotics

We start by looking at the definite operator A with $V_\gamma = \gamma V_0$ in more detail. Let $A^{D,N}$ denote the restrictions of the operator A to $\mathbb{R}_+$ with Dirichlet and Neumann boundary condition at zero, resp. By the spectral theorem, for symmetric potentials $V_\gamma(x)$

$$\mathrm{Spec}(A) = \mathrm{Spec}(A^D) \cup \mathrm{Spec}(A^N)$$

with account of multiplicities. Let $-\lambda_n^\#$ denote the eigenvalues of $A^\#$, $\# = D$ or N, ordered increasingly.

Sharp self-adjoint asymptotics (contd.)

Theorem (ML+Marcello Seri)

As $n \to \infty$,

$$\lambda_n^D(\gamma) = \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_1}(\gamma) + O\left(\frac{1}{n^2}\right)\right),$$
$$\lambda_n^N(\gamma) = \frac{\gamma^2}{4n^2}\left(1 - \frac{2}{\pi n}\Theta_{R_0}(\gamma) + O\left(\frac{1}{n^2}\right)\right),$$

where

$$R_k(\gamma) = \frac{J_k\left(2\sqrt{\gamma}\right)}{Y_k\left(2\sqrt{\gamma}\right)},$$

J_k *and* Y_k *are the Bessel functions of the first and second kind, resp., and* Θ_F *denotes the continuous branch of the multi-valued* $\operatorname{Arctan}(F(x))$ *such that* $\Theta_F(0) = 0$

Sharp non-self-adjoint asymptotics

Theorem

(i) *The eigenvalues of* H *lie asymptotically on the curves*

$$|\mathrm{Im}\,\mu| = \Upsilon(\gamma)\,|\,\mathrm{Re}\,\mu|^{3/2} + O\left((\mathrm{Re}\,\mu)^2\right), \qquad \mu \to 0$$

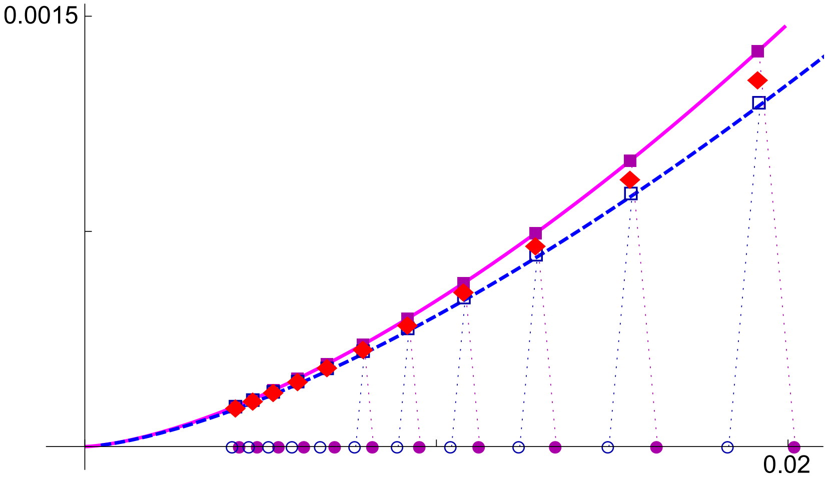
(ii) *More precisely, the eigenvalues* $\{\mu\}_n$ *of* H *in the first quadrant are related to the absolute values* $\lambda_n^{\#}$ *by*

$$\mu_n = \lambda_n^D + \Upsilon^-(\gamma)(\lambda_n^D)^{3/2} + O\left((\lambda_n^D)^2\right) = \lambda_n^N + \Upsilon^+(\gamma)(\lambda_n^N)^{3/2} + O(\dots)$$

as $n \to \infty$, *where* $\Upsilon^{\mp}(\gamma) = \frac{4}{\pi\gamma}\arctan\left(\frac{1}{\mathrm{i} \mp 2q(\gamma)}\right)$,

$$q(\gamma) := \pi\sqrt{\gamma}\left(J_0(2\sqrt{\gamma})\,J_1(2\sqrt{\gamma}) + Y_0(2\sqrt{\gamma})\,Y_1(2\sqrt{\gamma})\right).$$

Illustration



Comments

- Proof is based on a delicate result by Niko Temme (2015) on asymptotics of Kummer functions $U(-1/z, j, z)$, $z \to 0$, $j = 0, -1$.
- The general case seems to be hard, although the first step is trivial. If $\phi_\lambda(x)$ is a Jost solution of

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi(x) - V(x)\phi(x) = \lambda\phi(x) \qquad x \in \mathbb{R}_+,\ \lambda \in \mathbb{C}\setminus\mathbb{R}_+$$

  then the eigenvalues μ of the pencil are the zeros of

$$\phi'_\mu(0)\phi_{-\mu}(0) + \phi'_{-\mu}(0)\phi_\mu(0).$$

  To attack this asymptotically one should know sharp asymptotics of the M-function $\phi'_\mu(0)/\phi_\mu(0)$ of the Jost solution in the complex neighbourhood of $\mu = 0$, uniform in $\arg\mu$. The best results I know are due to Yafaev (in terms of scattering amplitude and phase) but...
  - they are not sharp enough, and
  - more importantly, they exclude important sectors $\{|\operatorname{Im}(\mu)| \le \varepsilon|\operatorname{Re}(\mu)|\}$ around the real axis.
- Any suggestions?

A matrix pencil

Let

$$A = \begin{pmatrix} b_1 & a_1 & & \\ a_1 & b_2 & a_2 & \\ & \ddots & \ddots & \ddots \\ & & & b_N \end{pmatrix}$$

be a symmetric Jacobi $N \times N$ matrix with real entries. When $a_j = 1$, A is usually referred to as a discrete Schrödinger operator, and the set of diagonal entries b_j as a potential. I am going to make everything even simpler and consider only constant potentials $b_j = c$, c is real, so my matrix becomes

$$A = A_{N;c} = \begin{pmatrix} c & 1 & 0 & \dots & 0 \\ 1 & c & 1 & \dots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \dots & 1 & c & 1 \\ 0 & \dots & 0 & 1 & c \end{pmatrix}$$

A matrix pencil (contd.)

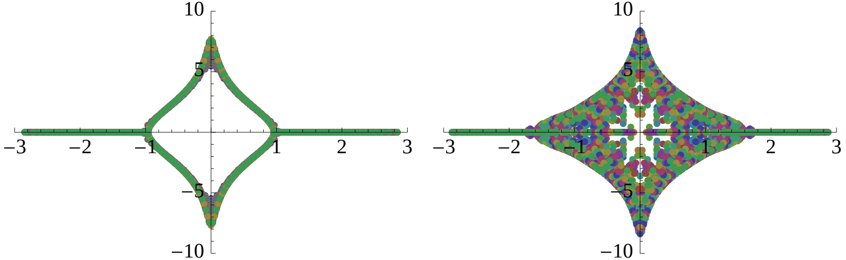
We consider an operator pencil $\mathcal{P}_{n,m;c} = A_{n+m;c} - \lambda B_{n,m}$ where

$$B = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}1\\ \ddots\\ 1\end{matrix}}\right\} m \text{ rows} \\ \left.\vphantom{\begin{matrix}-1\\ \ddots\\ -1\end{matrix}}\right\} n \text{ rows} \end{matrix}$$

We are interested mostly in the behaviour of eigenvalues when $m = n = N/2 \to \infty$.

A matrix pencil (contd.)

The problem gets even more complicated for general potentials, but can be reduced to studying equations involving *orthogonal polynomials* associated with a matrix A. However the question we need to study go, surprisingly, beyond the known results in the field. A typical spectral picture is like this:



We start with the following easy result on the localisation of eigenvalues of the pencil $\mathcal{P}_{n.m;c}$.

Theorem

(a) *The spectrum* $\operatorname{Spec}\mathcal{P}_{m,n;c}$ *is invariant under the symmetry* $\lambda \to \overline{\lambda}$, *and if* $m = n$ *also under the symmetry* $\lambda \to -\lambda$.

(b) *All the eigenvalues* $\lambda \in \operatorname{Spec}\mathcal{P}_{m,n;c}$ *satisfy*

$$|\lambda| < 2 + |c|.$$

(c) *If* $|c| \geq 2$, *then* $\operatorname{Spec}\mathcal{P}_{m,n;c} \subset \mathbb{R}$.

Rough localisation

Rough asymptotics of eigenvalues as $N \to \infty$ is given by

Theorem

The non-real eigenvalues of $\mathcal{P}_{m,n;c}$ converge uniformly to the real axis as $n, m \to \infty$. More precisely,

$$\begin{aligned} &\max\{|\operatorname{Im}(\lambda)| : \lambda \in \operatorname{Spec}\mathcal{P}_{m,n;c}\} \\ &\quad \le \max\left\{\frac{\log(m)}{m}(1+o(1)), \frac{\log(n)}{n}(1+o(1))\right\} \end{aligned} \tag{1}$$

as $m, n \to \infty$.

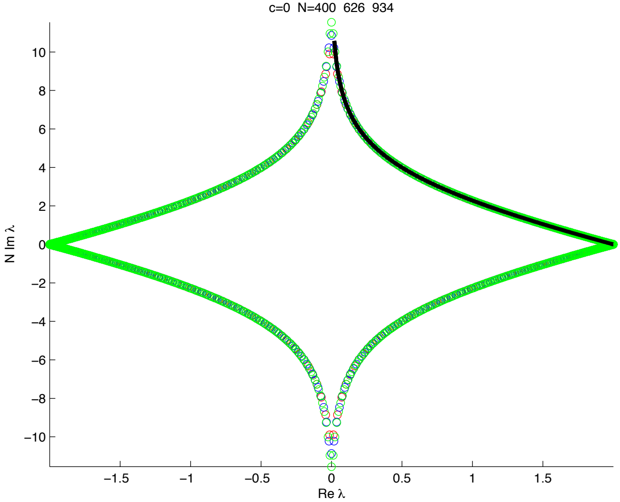
Note that the estimate is sharp in the following sense: it's attained, and it needs *both* $n, m \to \infty$.

The movie — the dependence on c

The movie and the code used to produce it are available in the Ancillary files section of http://arxiv.org/abs/1311.6741

Example, $c = 0$, $n = m = N/2$

Let us look at the simplest case $c = 0$.



Asymptotics, $c = 0$, $n = m = N/2$

Theorem

Let $c = 0$, $n = m = N/2 \to \infty$. The eigenvalues of $\mathcal{P}_{m,m;0}$ are all non-real, and those not lying on the imaginary axis satisfy

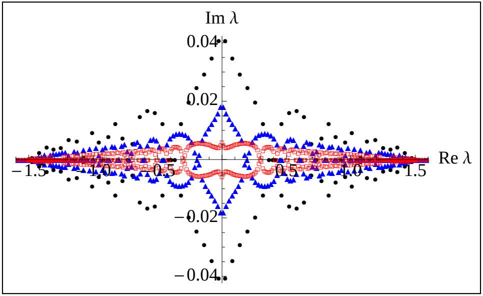
$$\operatorname{Im}(\lambda) = \pm \frac{Y_0(|\operatorname{Re}(\lambda)|)}{N} + o(N^{-1}),$$

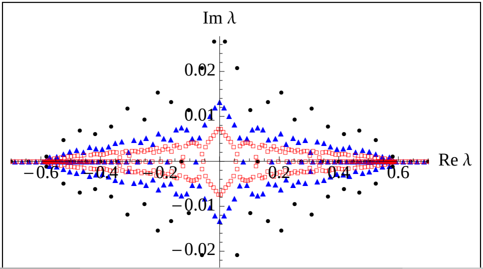
$$\operatorname{Re}(\lambda) = \pm 2\cos\left(\frac{2\pi k}{N+1}\right) + o(N^{-1}), \qquad k = 1, \dots, \left\lfloor \frac{m}{2} \right\rfloor,$$

$$Y_0(u) := \sqrt{4-u^2}\log\left(\tan\left(\frac{\pi}{4} + \frac{1}{2}\arccos\left(\frac{u}{2}\right)\right)\right).$$

If m is even, there are no other eigenvalues. If m is odd, there are additionally two purely imaginary eigenvalues at $\lambda = \pm \mathrm{i}\frac{\log(N/2)}{N/2}(1 + o(1))$.

Example, $c \neq 0$, $n = m = N/2$





Asymptotics, $c \neq 0$, $n = m = N/2$

Theorem

Let $0 < c < 2$, $n = m = N/2 \to \infty$. *The non-real eigenvalues of* $\mathcal{P}_{m,m;c}$ *satisfy*

$$|\operatorname{Im}(\lambda)| \leq \frac{Y_c(|\operatorname{Re}(\lambda)|)}{N} + o\left(N^{-1}\right),$$

where Y_c *is given by an explicit expression*

$$Y_c(u) := X_{c,u}^{-1}\left(\tan\left(\frac{1}{2}\arccos\left(\frac{u-c}{2}\right)\right)\tan\left(\frac{1}{2}\arccos\left(\frac{u+c}{2}\right)\right)\right),$$

and $X_{c,u}^{-1}$ *is the inverse of the monotonic increasing analytic function* $X_{c,u} : (0,\infty) \to (0,1)$ *defined by*

$$X_{c,u}(v) := \tanh\left(\frac{v}{2\sqrt{4-(u-c)^2}}\right)\tanh\left(\frac{v}{2\sqrt{4-(u+c)^2}}\right).$$

Idea of proof

Do not try to analyse directly a characteristic polynomial in λ.

Set $\lambda - c = z + 1/z$, $\lambda + c = w + 1/w$. Then for non-real eigenvalues

$$F_m(z)F_m(w) = -1,$$

where

$$F_m(z) = \frac{z^{m+1} - z^{-m-1}}{z^m - z^{-m}} = \frac{\sinh((m+1)\log z)}{\sinh(m \log z)} = \frac{U_m((z+1/z)/2)}{U_{m-1}((z+1/z)/2)}.$$

where U_n are the Chebyshev polynomials of the second kind, $U_n(\cos\theta) := \frac{\sin(n+1)\theta}{\sin\theta}$.

Idea of proof

Alas, numerous known results on asymptotics of ratios of Chebyshev polynomials (e.g. in Simon's two volumes) are not sharp enough! We need a good ansatz and some rather involved trickery with complex functions. Even this simple setup leaves a number of open questions, surprisingly simple to formulate but hard to prove. For example, asymptotic and numerical evidence suggest

Conjecture

$\lambda \in \operatorname{Spec} \mathcal{P}_{m,m;c} \setminus \mathbb{R}$ implies $|\lambda \pm c| < 2$.

This translates in terms of Chebyshev polynomials as

Conjecture

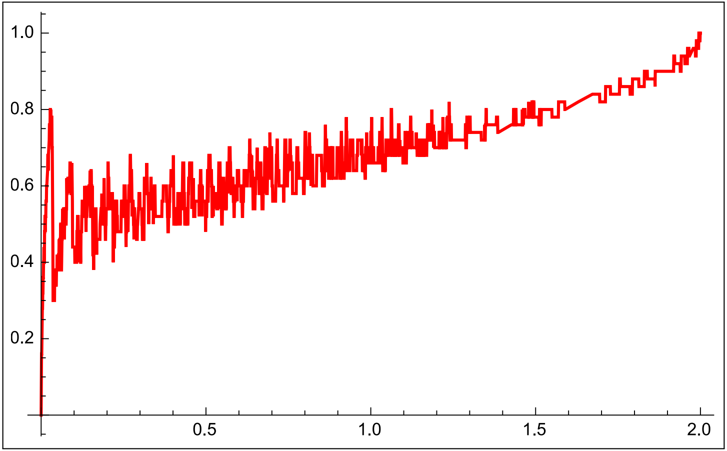
Let $\sigma, \tau \in \mathbb{C}$, $\operatorname{Im} \sigma = \operatorname{Im} \tau > 0$. If, for some $m \in \mathbb{N}$,

$$U_{m+1}(\sigma) U_{m+1}(\tau) + U_m(\sigma) U_m(\tau) = 0,$$

then $|\sigma| < 1$ and $|\tau| < 1$.

Proportion of real eigenvalues as function of c

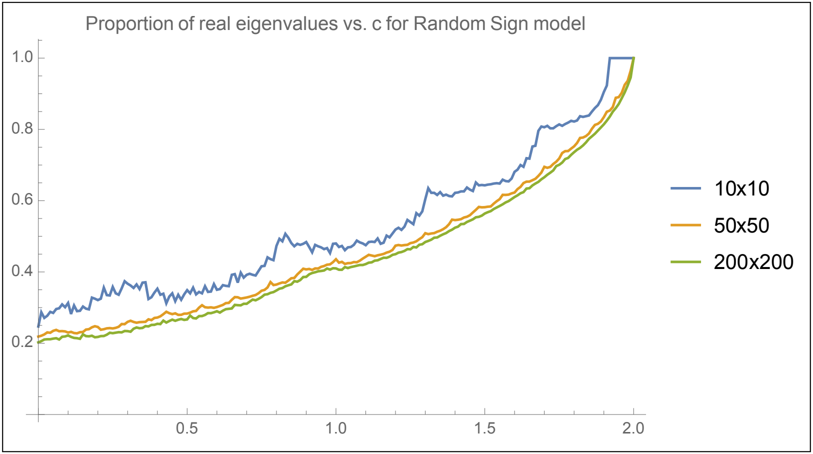
The problem is difficult in many ways!



There is a complicated interplay between the proportion of real eigenvalues and number-theoretical properties of some quantities including N and c.

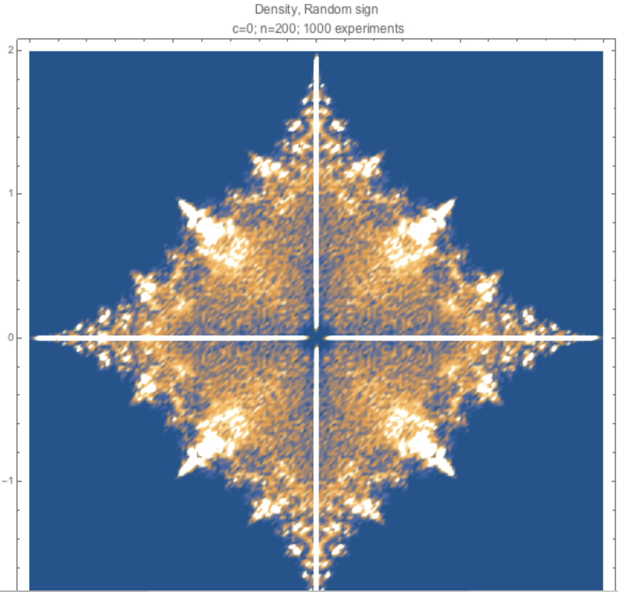
Random Sign model — real eigenvalues

Let us change the problem, and choose diagonal entires of B randomly to be ± 1 with probability $\frac{1}{2}$, and look at the proportion of real eigenvalues.

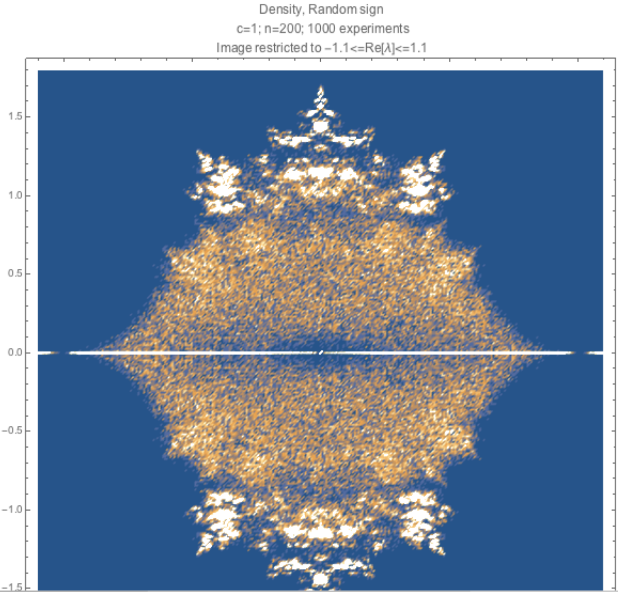


Any help from random matrix experts?

Random Sign model — complex eigenvalues, $c = 0$



Random Sign model — complex eigenvalues, $c = 1$



Random Sign model — complex eigenvalues, $c = 1.9$