Mathematical Analysis of High Aspect-Ratio Aircraft Wing in Subsonic Air Flow: Incompressible and Compressible Cases

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Flutter Analysis

- **Flutter is an instability endemic to aircraft that occurs at high enough airspeed in subsonic flight and sets a "flutter boundary" on attainable airspeed in the subsonic regime.**
  The determination of aircraft flutter characteristics is one of the most important safety aspects in the aircraft design and analysis process. Damage inflicted by flutter is extremely costly.

- **Flutter analysis is ranked by NASA aeronautics as one of the top research projects**

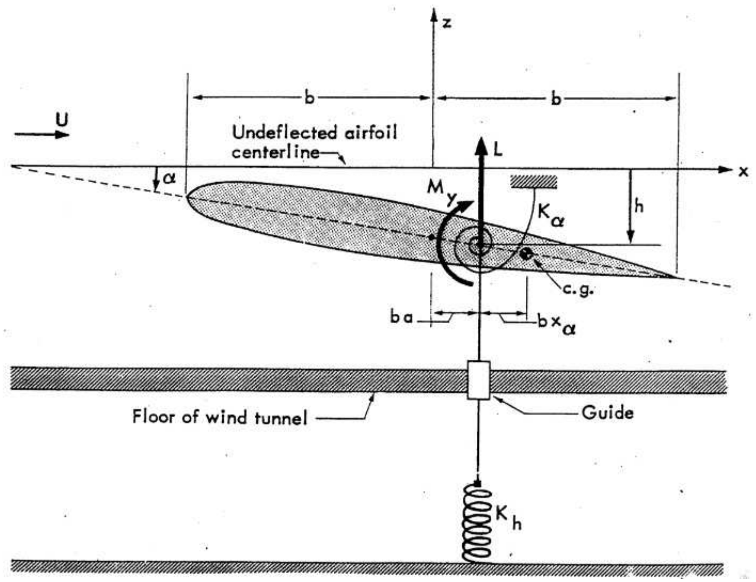
- **"Some fear flutter because they do not understand it" said the famous aerodynamist Theodore von Karman. "And some fear it," he added "because they do."**

"Flutter is the onset, beyond some speed -altitude combinations, of unstable and destructive vibrations of a lifting surface in an airstream. Flutter most commonly encountered on bodies subjected to large lateral aerodynamic loads of lift type, such as wings, tails, and control surfaces"

Theodorsen - Garrick experiment

$$\dot{\mathbf{\Psi}} = \mathbf{i}\mathcal{L}\mathbf{\Psi} + \int_{\mathbf{0}}^{\mathbf{t}} \mathcal{F}(\mathbf{t}-\sigma)\dot{\mathbf{\Psi}}(\sigma)\mathbf{d}\sigma$$

Evolution-convolution equation



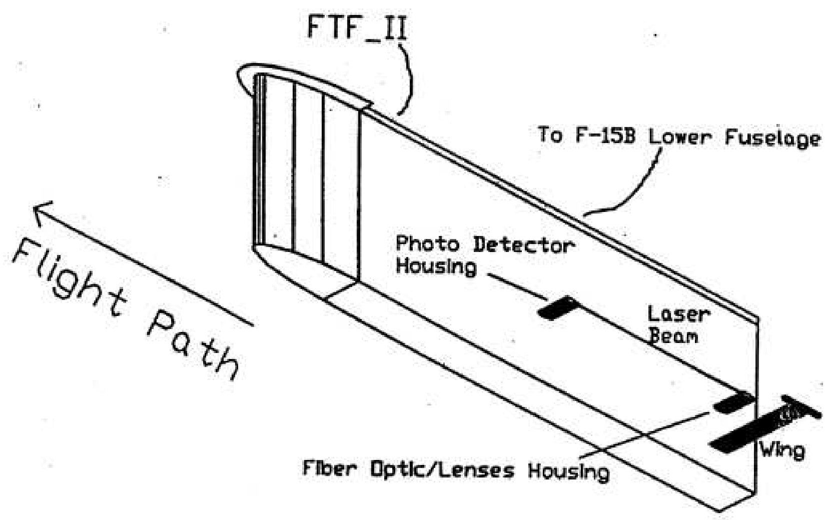
F-15B/FTF - II in Flight

NASA Dryden Flight Research Center Photo Collection
http://www.dfrc.nasa.gov/Gallery/Photo/index.html
NASA Photo: EC05–0028–18 Date: February 14, 2005 Photo By: Carla Thomas

All six divots of thermal insulation foam have been ejected from the flight test fixture on NASA's F–15B testbed as it returns from a LIFT experiment flight.

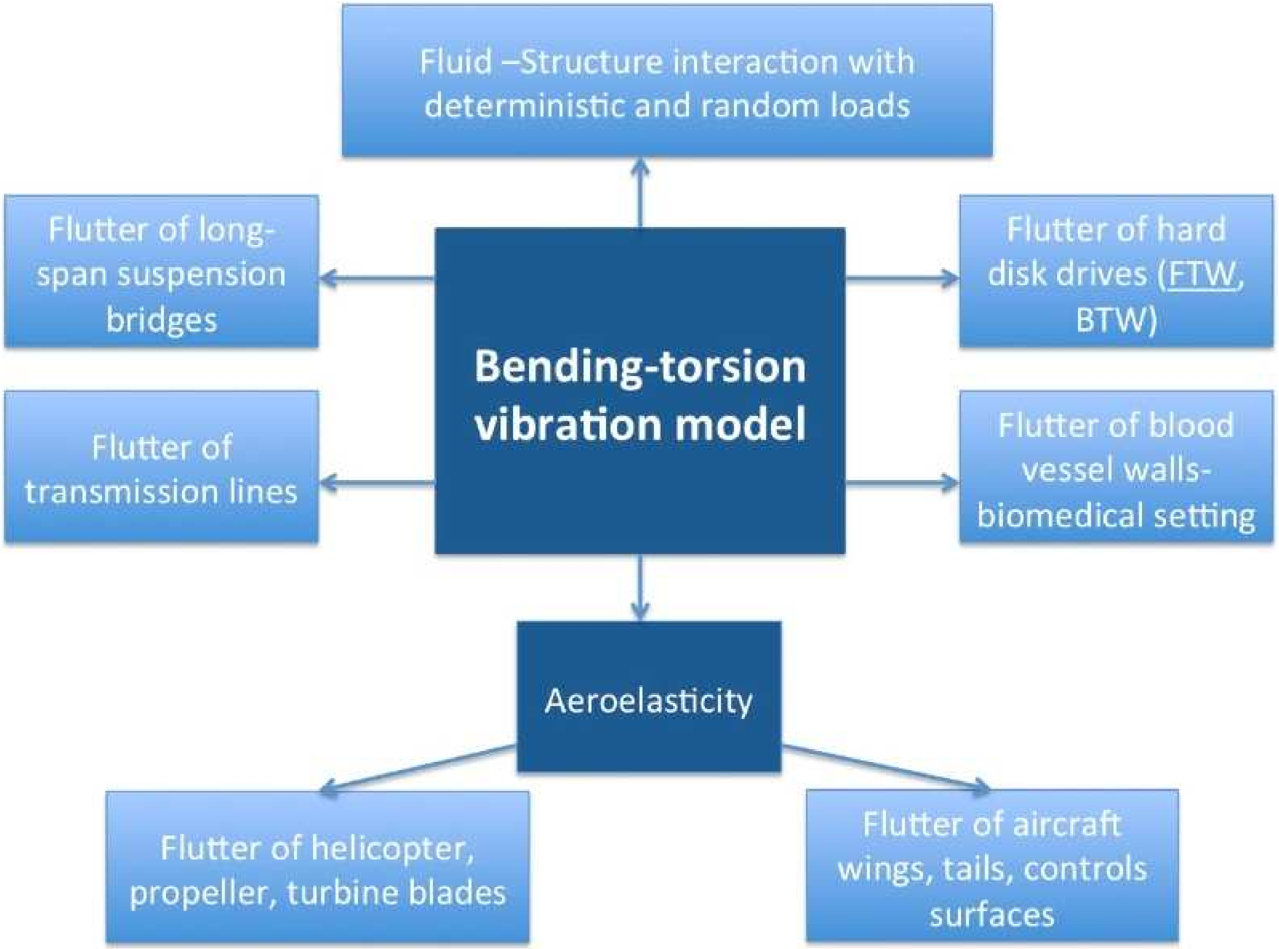
Flight Experiment Setup

- **Two Experiments conducted Simultaneously:**
  - Aeroelasticity Experiment with Flexible Wing
  - Testing of New Gust Monitoring Device



Some Facts about the flights

- **Taking into account all four flights, the experiment flew almost 7 hours.**
- **A total of 3.4 billion data points were sampled and recorded during the four flights by the high speed on-board recorder alone.**
- **Mach=0.8 @ 2000 feet were the maximum speed and altitudes achieved.**
- **Generated high angle of attack data & unique dynamic stall data $\Rightarrow$ sparked new research interest.**


Fluid –Structure interaction with deterministic and random loads
Flutter of long-span suspension bridges
Bending-torsion vibration model
Flutter of hard disk drives (FTW, BTW)
Flutter of transmission lines
Flutter of blood vessel walls-biomedical setting
Aeroelasticity
Flutter of helicopter, propeller, turbine blades
Flutter of aircraft wings, tails, controls surfaces

Models to be discussed

- **Model of high aspect-ratio aircraft wing in subsonic, inviscid, incompressible air flow**

- **Same type wing in subsonic, inviscid, compressible air flow. Aerodynamic loads and Possio integral equation**

Some of the main results to be presented

- **Explicit asymptotic formulas for ground vibrations modes and corresponding eigenfunctions**
- **Explicit asymptotic formulas for aeroelastic modes and corresponding mode shapes**
- **Analytic mechanism generating fluttering modes ("flutter matrix")**
- **Practical efficiency of explicit asymptotic formulas for aeroelastic modes; comparison with numerical results**

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Aeroelastic Problem

bending **torsion angle**

$$\mathbf{X}(\mathbf{x},\mathbf{t}) = (h(x,t),\ \alpha(x,t))^{\mathbf{T}}, \qquad -\mathbf{L} \le \mathbf{x} \le \mathbf{0},\ \mathbf{t} \ge \mathbf{0},$$

Model: system of two coupled damped integro-differential equations

$$(\mathbf{M_s} - \mathbf{M_a})\ddot{\mathbf{X}}(\mathbf{x},\mathbf{t}) + (\mathbf{D_s} - \mathbf{u}\mathbf{D_a})\dot{\mathbf{X}}(\mathbf{x},\mathbf{t}) +$$
$$(\mathbf{K_s} - \mathbf{u}^2\mathbf{K_a})\mathbf{X} = [\mathbf{f_1}(\mathbf{x},\mathbf{t}), \mathbf{f_2}(\mathbf{x},\mathbf{t})]^{\mathbf{T}}.$$

where $\mathbf{u} > 0$ **- stream velocity**

$$\mathbf{M_s} = \begin{bmatrix} \mathbf{m} & \mathbf{S} \\ \mathbf{S} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{M_a} = (-\pi\rho)\begin{bmatrix} \mathbf{1} & -\mathbf{a} \\ -\mathbf{a} & (\mathbf{a}^2 + \mathbf{1}/\mathbf{8}) \end{bmatrix},$$

$\mathbf{m}$ **- density of the structure,** $\mathbf{S}$ **- mass moment,**
$\mathbf{I}$ **- moment of inertia;** $\rho \ll 1$, **a** $\in [-1, 1]$;

$$\mathbf{D_s} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{D_a} = (-\pi\rho) \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{K_s} = \begin{bmatrix} \mathbf{E}\frac{\partial^4}{\partial \mathbf{x}^4} & \mathbf{0} \\ \mathbf{0} & -\mathbf{G}\frac{\partial^2}{\partial \mathbf{x}^2} \end{bmatrix}, \qquad \mathbf{K_a} = (-\pi\rho) \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

E - bending stiffness; G - torsion stiffness

$$\mathbf{f_1(x,t)} = -\mathbf{2}\pi\rho\int_{\mathbf{0}}^{\mathbf{t}}\left[\mathbf{uC_2(t}-\sigma) - \mathbf{\dot{C}_3(t}-\sigma)\right]\mathbf{g(x},\sigma)\mathbf{d}\sigma,$$

$$\mathbf{f_2(x,t)} = -\mathbf{2}\pi\rho\int_{\mathbf{0}}^{\mathbf{t}}\left[\mathbf{1/2C_1(t}-\sigma) - \mathbf{auC_2(t}-\sigma) + \mathbf{a\dot{C}_3(t}-\sigma) + \mathbf{uC_4(t}-\sigma) + \mathbf{1/2\dot{C}_5(t}-\sigma)\right]\mathbf{g(x},\sigma)\mathbf{d}\sigma$$

Aerodynamical functions: $C_i,\ i = 1\ldots 5$,

$$\mathbf{g(x},\sigma) = \mathbf{u}\,\dot{\alpha}(\mathbf{x},\sigma) + \mathbf{\ddot{h}(x},\sigma) + \left(\frac{\mathbf{1}}{\mathbf{2}} - \mathbf{a}\right)\ddot{\alpha}(\mathbf{x},\sigma)$$

Recursion Relations

$$\widehat{\mathbf{C}}_1(\lambda) = \int_0^\infty \mathbf{e}^{-\lambda \mathbf{t}} \mathbf{C}_1(\mathbf{t}) \mathbf{dt} = \frac{\mathbf{u}}{\lambda} \frac{\mathbf{e}^{-\lambda/\mathbf{u}}}{\mathbf{K}_0(\lambda/\mathbf{u}) + \mathbf{K}_1(\lambda/\mathbf{u})}, \quad \Re\lambda > \mathbf{0},$$

$$\mathbf{C}_2(\mathbf{t}) = \int_0^{\mathbf{t}} \mathbf{C}_1(\sigma) \mathbf{d}\sigma,$$

$$\mathbf{C}_3(\mathbf{t}) = \int_0^{\mathbf{t}} \mathbf{C}_1(\mathbf{t}-\sigma)(\mathbf{u}\sigma - \sqrt{\mathbf{u}^2\sigma^2 + 2\mathbf{u}\sigma}) \mathbf{d}\sigma,$$

$$\mathbf{C}_4(\mathbf{t}) = \mathbf{C}_2(\mathbf{t}) + \mathbf{C}_3(\mathbf{t}),$$

$$\mathbf{C}_5(\mathbf{t}) = \int_0^{\mathbf{t}} \mathbf{C}_1(\mathbf{t}-\sigma)((\mathbf{1}+\mathbf{u}\sigma)\sqrt{\mathbf{u}^2\sigma^2 + 2\mathbf{u}\sigma} - (\mathbf{1}+\mathbf{u}\sigma)^{\mathbf{2}}) \mathbf{d}\sigma,$$

$K_0(z)$ **and** $K_1(z)$ **- the modified Bessel functions (McDonald functions)**

Boundary Conditions

At the end $x = -L$:

$$\mathbf{h}(-\mathbf{L},\mathbf{t}) = \mathbf{h}'(-\mathbf{L},\mathbf{t}) = \alpha(-\mathbf{L},\mathbf{t}) = \mathbf{0}$$

At the end $x = 0$: self straining actuator action

$$\mathbf{E}\mathbf{h}''(\mathbf{0},\mathbf{t}) + \beta\dot{\mathbf{h}}'(\mathbf{0},\mathbf{t}) = \mathbf{0}, \qquad \mathbf{h}'''(\mathbf{0},\mathbf{t}) = \mathbf{0},$$

$$\mathbf{G}\alpha'(\mathbf{0},\mathbf{t}) + \delta\dot{\alpha}(\mathbf{0},\mathbf{t}) = \mathbf{0}, \qquad \beta,\delta \in \mathfrak{C}^{+} \cup \{\infty\},$$

If $\beta > 0$, then $\beta \equiv g_h$ - bending control gain,
If $\delta > 0$, then $\delta \equiv g_\alpha$ - torsion control gain

Initial Conditions

$$\mathbf{h}(\mathbf{x},\mathbf{0}) = \mathbf{h_0}(\mathbf{x}), \qquad \dot{\mathbf{h}}(\mathbf{x},\mathbf{0}) = \mathbf{h_1}(\mathbf{x}),$$
$$\alpha(\mathbf{x},\mathbf{0}) = \alpha_{\mathbf{0}}(\mathbf{x}), \qquad \dot{\alpha}(\mathbf{x},\mathbf{0}) = \alpha_{\mathbf{1}}(\mathbf{x})$$

Assumptions

(a)

$$\mathbf{det} \begin{bmatrix} \mathbf{m} & \mathbf{S} \\ \mathbf{S} & \mathbf{I} \end{bmatrix} > \mathbf{0},$$

(b)

$$\mathbf{0} < \mathbf{u} \leq \frac{\sqrt{\mathbf{2G}}}{\mathbf{L}\sqrt{\pi\rho}} \quad \textbf{(divergence speed)}$$

Energy of vibration

$$\mathbf{E(t)} = \frac{1}{2}\int_{-\mathbf{L}}^{\mathbf{0}} [\mathbf{E}|\mathbf{h}''(\mathbf{x},\mathbf{t})|^{\mathbf{2}} + \mathbf{G}|\alpha'(\mathbf{x},\mathbf{t})|^{\mathbf{2}} + \widetilde{\mathbf{m}}|\dot{\mathbf{h}}(\mathbf{x},\mathbf{t})|^{\mathbf{2}} +$$

$$\tilde{\mathbf{I}}|\dot{\alpha}(\mathbf{x},\mathbf{t})|^{\mathbf{2}} + \widetilde{\mathbf{S}}(\dot{\alpha}\overline{\dot{\mathbf{h}}} + \overline{\dot{\alpha}}\dot{\mathbf{h}}) - \pi\rho\mathbf{u}^{\mathbf{2}}|\alpha(\mathbf{x},\mathbf{t})|^{\mathbf{2}}]\mathbf{dx}$$

where $\tilde{\mathbf{I}} = \mathbf{I} + \pi\rho(\mathbf{a}^{\mathbf{2}} + \mathbf{1/8})$, $\widetilde{\mathbf{m}} = \mathbf{m} - \pi\rho$, $\tilde{\mathbf{S}} = \mathbf{S} - \pi\rho\mathbf{a}$

Energy dissipation

$$\dot{\mathbf{E}}(\mathbf{t}) = -\mathbf{E}|\dot{\mathbf{h}}'(\mathbf{0},\mathbf{t})|^{\mathbf{2}}\,\Re\,\beta - \mathbf{G}|\dot{\alpha}(\mathbf{0},\mathbf{t})|^{\mathbf{2}}\Re\,\delta$$

Initial - Boundary Value Problem

$$\begin{cases} \tilde{\mathbf{m}}\ddot{\mathbf{h}} + \tilde{\mathbf{S}}\ddot{\alpha} + \mathbf{E}\mathbf{h}'''' + \pi\rho\mathbf{u}\dot{\alpha} & = & \mathbf{f_1}(\mathbf{x},\mathbf{t}), & -\mathbf{L} \le \mathbf{x} \le \mathbf{0}, \\ \tilde{\mathbf{S}}\ddot{\mathbf{h}} + \tilde{\mathbf{I}}\ddot{\alpha} - \mathbf{G}\alpha'' + \pi\rho\mathbf{u}^2\alpha - \pi\rho\mathbf{u}\mathbf{h} & = & \mathbf{f_2}(\mathbf{x},\mathbf{t}), & \mathbf{t} > \mathbf{0} \end{cases}$$

Boundary Conditions

at $\mathbf{x} = -\mathbf{L}$**:** $\mathbf{h}(-\mathbf{L},\mathbf{t}) = \mathbf{h}'(-\mathbf{L},\mathbf{t}) = \alpha(-\mathbf{L},\mathbf{t}) = \mathbf{0}$

at $\mathbf{x} = \mathbf{0}$**:** $\mathbf{h}'''(\mathbf{0}) = \mathbf{0}$;

$$\begin{cases} \mathbf{E}\mathbf{h}''(\mathbf{0},\mathbf{t}) + \beta\dot{\mathbf{h}}'(\mathbf{0},\mathbf{t}) = \mathbf{0} \\ \mathbf{G}\alpha'(\mathbf{0},\mathbf{t}) + \delta\dot{\alpha}(\mathbf{0},\mathbf{t}) = \mathbf{0} \end{cases}$$

Initial Conditions

$$\mathbf{h}(\mathbf{x},\mathbf{0}) = \mathbf{h_0}, \quad \dot{\mathbf{h}}(\mathbf{x},\mathbf{0}) = \mathbf{h_1}, \quad \alpha(\mathbf{x},\mathbf{0}) = \alpha_{\mathbf{0}}, \quad \dot{\alpha}(\mathbf{x},\mathbf{0}) = \alpha_{\mathbf{1}}$$

Operator Setting

State space ↔ Energy space

$\mathcal{H}$ **- set of 4 component vector - valued functions**

$$\Psi = (h, \dot{h}, \alpha, \dot{\alpha})^T \equiv (\psi_0, \psi_1, \psi_2, \psi_3)^T$$

satisfying

$$\psi_0(-L) = \psi_0'(-L) = \psi_2(-L) = 0$$

Norm of state space (Hilbert space):

$$||\mathbf{\Psi}||^2_{\mathcal{H}} = \mathbf{1/2} \int_{-\mathbf{L}}^{\mathbf{0}} \Big[\mathbf{E}|\psi_{\mathbf{0}}''(\mathbf{x})|^{\mathbf{2}} + \mathbf{G}|\psi_{\mathbf{2}}'(\mathbf{x})|^{\mathbf{2}} + \widetilde{\mathbf{m}}|\psi_{\mathbf{1}}(\mathbf{x})|^{\mathbf{2}} + \tilde{\mathbf{I}}|\psi_{\mathbf{3}}(\mathbf{x})|^{\mathbf{2}}$$

$$+ \tilde{\mathbf{S}}(\psi_{\mathbf{3}}(\mathbf{x})\bar{\psi}_{\mathbf{1}}(\mathbf{x}) + \bar{\psi}_{\mathbf{3}}(\mathbf{x})\psi_{\mathbf{1}}(\mathbf{x})) - \pi\rho\mathbf{u}^{\mathbf{2}}|\psi_{\mathbf{2}}(\mathbf{x})|^{\mathbf{2}} \Big] \, \mathbf{dx}$$

$$\dot{\mathbf{\Psi}} = \mathbf{i}\mathcal{L}\mathbf{\Psi} + \int_{\mathbf{0}}^{\mathbf{t}} \mathcal{F}(\mathbf{t}-\sigma)\dot{\mathbf{\Psi}}(\sigma)\mathbf{d}\sigma$$

First order in time "evolution - convolution" equation in $\mathcal{H}$

$$\dot{\Psi} = i\mathcal{L}_{\beta\delta}\Psi + \tilde{\mathcal{F}}\dot{\Psi}, \quad \Psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T, \quad \Psi|_{t=0} = \Psi_0$$

$\mathcal{L}_{\beta\delta}$ - **matrix differential operator in $\mathcal{H}$**
$\widetilde{\mathcal{F}}$ - **matrix integral operator in $\mathcal{H}$**

$\mathcal{L}_{\beta\delta}$ - **matrix differential operator in** $\mathcal{H}$

$$\mathcal{L}_{\beta\delta} = -\mathbf{i}\begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ -\frac{\mathbf{E\tilde{I}}}{\boldsymbol{\Delta}}\frac{\mathbf{d^4}}{\mathbf{dx^4}} & -\frac{\pi\rho\mathbf{u\tilde{S}}}{\boldsymbol{\Delta}} & -\frac{\mathbf{\tilde{S}}}{\boldsymbol{\Delta}}\left(\mathbf{G}\frac{\mathbf{d^2}}{\mathbf{dx^2}} + \pi\rho\mathbf{u^2}\right) & -\frac{\pi\rho\mathbf{u\tilde{I}}}{\boldsymbol{\Delta}} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \frac{\mathbf{E\tilde{S}}}{\boldsymbol{\Delta}}\frac{\mathbf{d^4}}{\mathbf{dx^4}} & \frac{\pi\rho\mathbf{u\widetilde{m}}}{\boldsymbol{\Delta}} & \frac{\widetilde{\mathbf{m}}}{\boldsymbol{\Delta}}\left(\mathbf{G}\frac{\mathbf{d^2}}{\mathbf{dx^2}} + \pi\rho\mathbf{u^2}\right) & \frac{\pi\rho\mathbf{u\tilde{S}}}{\boldsymbol{\Delta}} \end{bmatrix}$$

defined on the domain

$$\begin{aligned} \mathcal{D}(\mathcal{L}_{\beta\delta}) \quad = \quad \{\boldsymbol{\Psi} \in \mathcal{H} : \ & \psi_\mathbf{0} \in \mathbf{H^4}(-\mathbf{L}, \mathbf{0}),\ \psi_\mathbf{1} \in \mathbf{H^2}(-\mathbf{L}, \mathbf{0}), \\ & \psi_\mathbf{2} \in \mathbf{H^2}(-\mathbf{L}, \mathbf{0}),\ \psi_\mathbf{3} \in \mathbf{H^1}(-\mathbf{L}, \mathbf{0}); \\ & \psi_\mathbf{1}(-\mathbf{L}) = \psi_\mathbf{1}'(-\mathbf{L}) = \psi_\mathbf{3}(-\mathbf{L}) = \mathbf{0};\ \ \psi_\mathbf{0}'''(\mathbf{0}) = \mathbf{0}; \\ & \mathbf{E}\psi_\mathbf{0}''(\mathbf{0}) + \beta\psi_\mathbf{1}'(\mathbf{0}) = \mathbf{0},\ \ \mathbf{G}\psi_\mathbf{2}'(\mathbf{0}) + \delta\psi_\mathbf{3}(\mathbf{0}) = \mathbf{0}\} \end{aligned}$$

$$\boldsymbol{\Delta} = \widetilde{\mathbf{m}}\mathbf{\tilde{I}} - \mathbf{\tilde{S}^2} > \mathbf{0}$$

$\widetilde{\mathcal{F}}$ - matrix integral operator in $\mathcal{H}$

$$\widetilde{\mathcal{F}} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\tilde{\mathbf{I}}(\tilde{\mathbf{C}}_{\mathbf{1}}*) - \tilde{\mathbf{S}}(\tilde{\mathbf{C}}_{\mathbf{2}}*)] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [-\tilde{\mathbf{S}}(\tilde{\mathbf{C}}_{\mathbf{1}}*) + \widetilde{\mathbf{m}}(\tilde{\mathbf{C}}_{\mathbf{2}}*)] \end{bmatrix} \times$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{u} & (\mathbf{1/2} - \mathbf{a}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{u} & (\mathbf{1/2} - \mathbf{a}) \end{bmatrix}$$

Spectral properties of both the differential operator $\mathcal{L}_{\beta\delta}$ and the integral operator $\widetilde{\mathcal{F}}$ are of crucial importance for the representation of the solution

$$\dot{\Psi} = i\mathcal{L}_{\beta\delta}\Psi + \widetilde{\mathcal{F}}\dot{\Psi}, \quad \Psi|_{t=0} = \Psi_0$$

Laplace transform representation for solution:

$$\widehat{\Psi}(\lambda) = \left(\lambda I - i\mathcal{L}_{\beta\delta} - \lambda\widehat{\mathcal{F}}(\lambda)\right)^{-1} (I - \widehat{\mathcal{F}}(\lambda))\Psi_0$$

Goal: find the solution in space - time domain, i.e., "calculate" the inverse Laplace transform of $\widehat{\Psi}$

$$\mathcal{R}(\lambda) = \left(\lambda I - i\mathcal{L}_{\beta\delta} - \lambda\widehat{\mathcal{F}}(\lambda)\right)^{-1}$$

Generalized resolvent operator $\Rightarrow$
analytic operator - valued function of λ on the complex plane having a branch - cut along the negative real semi - axis.

Poles of $\mathcal{R}(\lambda)$ - discrete spectrum $\leftrightarrow$ aeroelastic modes,
Branch cut - continuous spectrum

Spectral asymptotics of $\mathcal{L}_{\beta\delta}$

Two - branch spectrum: If $|\delta| \neq \sqrt{\mathbf{G}\tilde{\mathbf{I}}}$, then

β- branch

$$\mu_{\mathbf{n}}^{\beta} = (\operatorname{sgn}\mathbf{n})\pi^{2}/\mathbf{L}^{2}\sqrt{\mathbf{E}\tilde{\mathbf{I}}/\boldsymbol{\Delta}}\ (\mathbf{n}-\mathbf{1}/\mathbf{4})^{\mathbf{2}} + \kappa_{\mathbf{n}}(\omega),$$
$$\omega = |\delta|^{-\mathbf{1}} + |\beta|^{-\mathbf{1}},\quad |\mathbf{n}| \to \infty,$$

where $\boldsymbol{\Delta} = \widetilde{\mathbf{m}}\tilde{\mathbf{I}} - \tilde{\mathbf{S}}^{\mathbf{2}}$, and

$$\sup_{\mathbf{n}\in\mathbb{Z}}\{|\kappa_{\mathbf{n}}(\omega)|\} = \mathbf{C}(\omega),\quad \mathbf{C}(\omega) \longrightarrow \mathbf{0} \text{ as } \omega \longrightarrow \mathbf{0}$$

δ-branch

$$\mu_{\mathbf{n}}^{\delta} = \frac{\pi\mathbf{n}}{\mathbf{L}\sqrt{\tilde{\mathbf{I}}/\mathbf{G}}} + \frac{\mathbf{i}}{\mathbf{2L}\sqrt{\tilde{\mathbf{I}}/\mathbf{G}}}\ \ln\frac{\delta + \sqrt{\mathbf{G}\tilde{\mathbf{I}}}}{\delta - \sqrt{\mathbf{G}\tilde{\mathbf{I}}}} + \mathbf{O}(|\mathbf{n}|^{-\mathbf{1}/\mathbf{2}})$$

Definition: $\exists$ **a nontrivial** $\mathbf{\Phi_n}$ **such that**

$$\left(\lambda_n I - i\mathcal{L}_{\beta\delta} - \lambda_n \widehat{F}(\lambda_n)\right)\Phi_n = 0,$$

$\Rightarrow \boldsymbol{\lambda_n}$**- aeroelastic mode,** $\mathbf{\Phi_n}$**- mode shape**

Theorem 1.

The set of aeroelastic modes $\{\boldsymbol{\lambda_n}\}$ **is asymptotically close to the set** $\{\mathbf{i}\boldsymbol{\mu_n}\}$ **where** $\{\boldsymbol{\mu_n}\}$ **are eigenvalues of** $\mathcal{L}_{\beta\delta}$

Theorem 2.

a) $\mathcal{L}_{\beta\delta}$ **is a closed linear operator with compact resolvent;**
b) $\mathcal{L}_{\beta\delta}$ **is nonselfadjoint unless** $\Re\beta = \Re\delta = 0$**;**
c) **If** $\Re\beta > 0$ **and** $\Re\delta > 0$**, then** $\mathcal{L}_{\beta\delta}$ **- dissipative:**

$$\Im(\mathcal{L}_{\beta\delta}\Psi, \Psi)_{\mathcal{H}} \geq 0, \quad \Psi \in \mathcal{D}(\mathcal{L}_{\beta\delta});$$

d) **Adjoint operator** $\mathcal{L}^*_{\beta\delta}$**,** $\quad \beta, \delta \mapsto -\bar{\beta}, -\bar{\delta}$

Analytical mechanism generating fluttering modes

Structure of matrix integral operator

$$\mathfrak{S}(\lambda) = \lambda \mathbf{I} - \mathbf{i}\mathcal{L}_{\beta\delta} - \lambda\widehat{\mathcal{F}}(\lambda)$$

$$\widehat{\mathcal{F}} \Rightarrow \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\tilde{\mathbf{I}}(\tilde{\mathbf{C}}\mathbf{1}*) - \tilde{\mathbf{S}}(\tilde{\mathbf{C}}_\mathbf{2}*)] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & [-\tilde{\mathbf{S}}(\tilde{\mathbf{C}}_\mathbf{1}*) + \tilde{\mathbf{m}}(\tilde{\mathbf{C}}_\mathbf{2}*)] \end{bmatrix} \times$$

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{u} & (\mathbf{1/2} - \mathbf{a}) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{u} & (\mathbf{1/2} - \mathbf{a}) \end{bmatrix}, \quad \mathbf{\Delta} = \tilde{\mathbf{m}}\tilde{\mathbf{I}} - \tilde{\mathbf{S}}^\mathbf{2}$$

$\widehat{\mathcal{F}}(\lambda)$ is a Laplace transform of $\mathcal{F}(\mathbf{t}) \Rightarrow$ (kernel of convolution operator)

$$\widehat{\mathcal{F}}(\lambda) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{L}(\lambda) & \mathbf{u}\mathcal{L}(\lambda) & (\mathbf{1/2-a})\mathcal{L}(\lambda) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{N}(\lambda) & \mathbf{u}\mathcal{N}(\lambda) & (\mathbf{1/2-a})\mathcal{N}(\lambda) \end{bmatrix}$$

$$\mathcal{L}(\lambda) = -\frac{\mathbf{2}\pi\rho\mathbf{u}}{\lambda\mathbf{\Delta}}\left\{-\tilde{\mathbf{S}}/\mathbf{2} + [\tilde{\mathbf{I}} + (\mathbf{1/2+a})\tilde{\mathbf{S}}]\mathbf{T}(\lambda/\mathbf{u})\right\},$$

$$\mathcal{N}(\lambda) = -\frac{\mathbf{2}\pi\rho\mathbf{u}}{\lambda\mathbf{\Delta}}\left\{-\widetilde{\mathbf{m}}/\mathbf{2} + [\tilde{\mathbf{S}} + (\mathbf{1/2+a})\widetilde{\mathbf{m}}]\mathbf{T}(\lambda/\mathbf{u})\right\}$$

Theodorsen function:

$$\mathbf{T}(\mathbf{z}) = \frac{\mathbf{K_1}(\mathbf{z})}{\mathbf{K_0}(\mathbf{z}) + \mathbf{K_1}(\mathbf{z})}$$

$\mathbf{K_0}$, $\mathbf{K_1}$ **the modified Bessel functions**
$\mathbf{K_n}(\mathbf{z}) = \mathbf{1/2\pi i e^{\pi n i/2} H_n^{(1)}(iz)}$

Definitions:

$$\mathbf{K_0}(\mathbf{z}) = \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \frac{\mathbf{z^{2m}}}{\mathbf{2^{2m}(m!)^2}} \left(\psi(\mathbf{m}+\mathbf{1}) - \frac{\mathbf{1}}{\mathbf{m}+\mathbf{1}} \ln(\mathbf{z}/\mathbf{2})\right),$$

$$\mathbf{K_1}(\mathbf{z}) = \frac{\mathbf{1}}{\mathbf{z}} + \frac{\mathbf{z}}{\mathbf{2}} \sum_{\mathbf{m}=\mathbf{0}}^{\infty} \frac{\mathbf{z^{2m}}}{\mathbf{2^{2m}(m!)^2(m+1)}} \{\ln(\mathbf{z}/\mathbf{2}) - \tfrac{\mathbf{1}}{\mathbf{2}}[\psi(\mathbf{m}+\mathbf{1}) - \psi(\mathbf{m}+\mathbf{2})]\},$$

where $\psi(\mathbf{m})$ **is the digamma function**

Asymptotics as $|\mathbf{z}| \to \infty$

$$\mathbf{T(z) = 1/2 + 1/16z + O(z^{-2})}$$

$$\mathbf{V(z) = T(z) - 1/2 \to 0 \quad as\ |z| \to \infty}$$

$$\mathfrak{S}(\lambda) = \lambda \mathbf{I} - \mathbf{i}\mathcal{L}_{\beta\delta} - \lambda\widehat{\mathcal{F}}(\lambda)$$

"Flutter Matrix"

$$\lambda \widehat{\mathcal{F}}(\lambda) = \mathfrak{M} + \mathfrak{N}(\lambda),$$

where

$$\mathfrak{M} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{uA} & (\mathbf{1/2} - \mathbf{a})\mathbf{A} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} & \mathbf{uB} & (\mathbf{1/2} - \mathbf{a})\mathbf{B} \end{bmatrix}$$

$$\mathbf{A} = -\pi\rho\mathbf{u}/\boldsymbol{\Delta}[\tilde{\mathbf{I}} + (\mathbf{a} - \mathbf{1/2})\tilde{\mathbf{S}}],$$
$$\mathbf{B} = \pi\rho\mathbf{u}/\boldsymbol{\Delta}[\tilde{\mathbf{S}} + (\mathbf{a} - \mathbf{1/2})\widetilde{\mathbf{m}}]$$

$$\mathfrak{N}(\lambda) = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A_1}(\lambda) & \mathbf{uA_1}(\lambda) & (\mathbf{1/2-a})\mathbf{A_1}(\lambda) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B_1}(\lambda) & \mathbf{uB_1}(\lambda) & (\mathbf{1/2-a})\mathbf{B_1}(\lambda) \end{bmatrix}$$

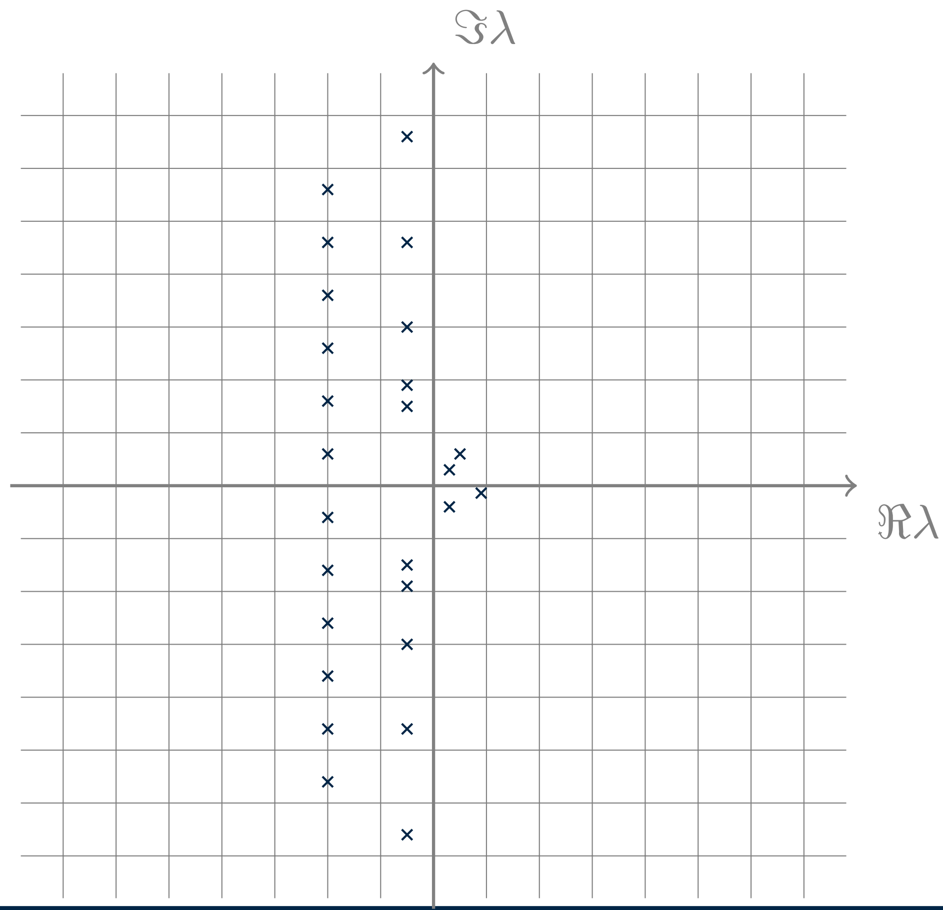
$$\mathbf{A_1}(\lambda) = -\mathbf{2}\pi\rho\mathbf{u}\boldsymbol{\Delta}^{-\mathbf{1}}\mathbf{V}(\lambda/\mathbf{u})[\tilde{\mathbf{I}} + (\mathbf{a}+\mathbf{1/2})\tilde{\mathbf{S}}],$$

$$\mathbf{B_1}(\lambda) = \mathbf{2}\pi\rho\mathbf{u}\boldsymbol{\Delta}^{-\mathbf{1}}\mathbf{V}(\lambda/\mathbf{u})[\tilde{\mathbf{S}} + (\mathbf{a}+\mathbf{1/2})\widetilde{\mathbf{m}}]$$

$$\|\mathfrak{N}(\lambda)\| \le C|\lambda^{-1}|, \qquad |\lambda| \to \infty$$

Numerical results for the eigenvalues of the operator $i\mathcal{L}_{\beta\delta} + \mathfrak{M}$

Recall: $\mathcal{R}(\lambda) = (\lambda I - i\mathcal{L}_{\beta\delta} - \mathfrak{M} - \mathfrak{N}(\lambda))^{-1}$
$\mathfrak{N}(\lambda)$-**Asymptotically small**
$\mathfrak{M}$-**"Flutter matrix"**



$$\mathfrak{S}(\lambda) = \lambda \mathbf{I} - \mathbf{i}\mathcal{L}_{\beta\delta} - \mathfrak{M} - \mathfrak{N}(\lambda), \quad \mathcal{R}(\lambda) = \mathfrak{S}(\lambda)^{-\mathbf{1}}$$

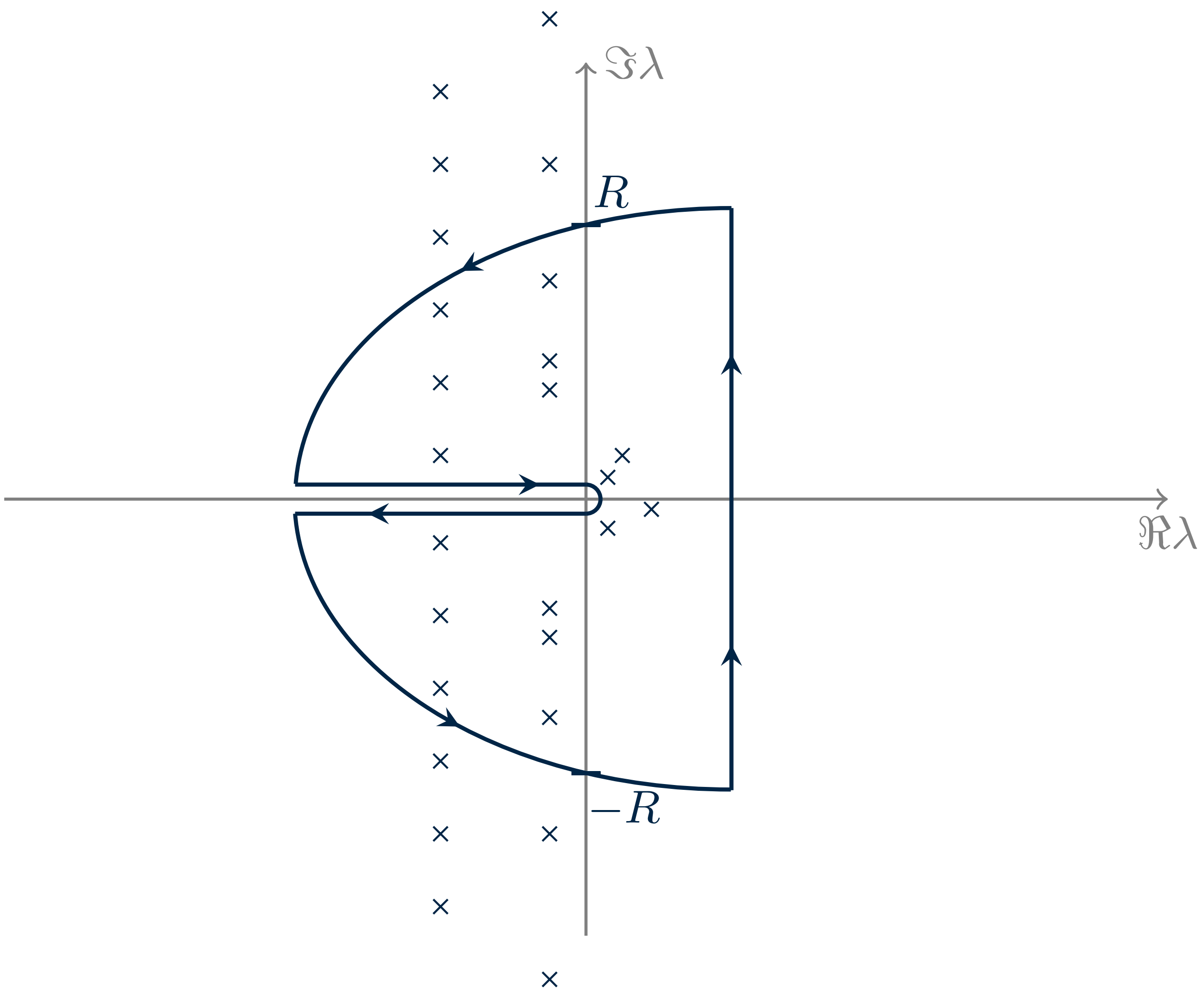
Spectrum: $\{\lambda_{\mathbf{n}}^{\beta}\}_{\mathbf{n}\in\mathbb{Z}} \cup \{\lambda_{\mathbf{n}}^{\delta}\}_{\mathbf{n}\in\mathbb{Z}}$ **(aeroelastic modes)**

Mode shapes: $\{\boldsymbol{\Phi}_{\mathbf{n}}^{\beta}\}_{\mathbf{n}\in\mathbb{Z}} \cup \{\boldsymbol{\Phi}_{\mathbf{n}}^{\delta}\}_{\mathbf{n}\in\mathbb{Z}}$

Properties of mode shapes

1. **Minimality**
2. **Completeness in H**
3. **Riesz basis property**

Contour Integration



$$\begin{aligned}\mathbf{\Psi}(\mathbf{x},\mathbf{t}) = \quad & \sum_{\mathbf{n}\in\mathbb{Z}'} \mathbf{e}^{\lambda_{\mathbf{n}}^{\beta}\mathbf{t}} \left(\left(\mathbf{I} - \hat{\mathcal{F}}(\lambda_{\mathbf{n}}^{\beta})(\lambda_{\mathbf{n}}^{\beta}) \right) \mathbf{\Psi_0}, \mathbf{\Phi}_{\mathbf{n}}^{\beta *} \right) \mathbf{\Phi}_{\mathbf{n}}^{\beta} + \\ & \sum_{\mathbf{n}\in\mathbb{Z}'} \mathbf{e}^{\lambda_{\mathbf{n}}^{\delta}\mathbf{t}} \left(\left(\mathbf{I} - \hat{\mathcal{F}}(\lambda_{\mathbf{n}}^{\delta})(\lambda_{\mathbf{n}}^{\delta}) \right) \mathbf{\Psi_0}, \mathbf{\Phi}_{\mathbf{n}}^{\delta *} \right) \mathbf{\Phi}_{\mathbf{n}}^{\delta} + \\ & \frac{\mathbf{1}}{\pi} \int_{\mathbf{0}}^{\infty} \mathbf{e}^{-\mathbf{rt}} \left(\Im \mathcal{R}(-\mathbf{r}) \left(\mathbf{I} - \hat{\mathcal{F}}(-\mathbf{r}) \right) \mathbf{\Psi_0} \right) \mathbf{dr}\end{aligned}$$

"Circle of Instability"

Main model equation: $\dot{\Psi}(t) = i\left(\mathcal{L}_{\beta\delta} - i\mathfrak{M}\right)\Psi(t) + \int\limits_0^t G(t-\tau)\Psi(\tau)d\tau$

Reduced model equation: $\dot{\Psi}(t) = i\mathcal{K}_{\beta\delta}\Psi(t), \quad \mathcal{K}_{\beta\delta} = \mathcal{L}_{\beta\delta} - i\mathfrak{M}.$

$$\widetilde{\mathbf{m}}\ddot{\mathbf{h}} + \tilde{\mathbf{S}}\ddot{\alpha} + \mathbf{E}\mathbf{h}'''' - \pi\rho\mathbf{u}\dot{\mathbf{h}} + \pi\rho\mathbf{u}\left(\tfrac{\mathbf{3}}{\mathbf{2}} - \mathbf{a}\right)\dot{\alpha} + \pi\rho\mathbf{u}^{\mathbf{2}}\alpha = \mathbf{0}$$
$$\widetilde{\mathbf{S}}\ddot{\mathbf{h}} + \widetilde{\mathbf{I}}\ddot{\alpha} - \mathbf{G}\alpha'' - \pi\rho\mathbf{u}\left(\tfrac{\mathbf{1}}{\mathbf{2}} + \mathbf{a}\right)\dot{\mathbf{h}} + \pi\rho\mathbf{u}\left(\tfrac{\mathbf{1}}{\mathbf{2}} - \mathbf{a}\right)^{\mathbf{2}}\dot{\alpha} - \pi\rho\mathbf{u}^{\mathbf{2}}\left(\tfrac{\mathbf{1}}{\mathbf{2}} + \mathbf{a}\right)\alpha = \mathbf{0}$$

Heuristic derivation of the energy functional accounting air-structure interaction;
Non-local in time (or memory -type) functional

University of New Hampshire

$$\mathbf{E(t)} = \frac{1}{2}\int_{-L}^{0}\Big\{\mathbf{E}|\mathbf{h}''|^2 + \mathbf{G}|\alpha'|^2 + \widetilde{\mathbf{m}}|\dot{\mathbf{h}}|^2 + \widetilde{\mathbf{I}}|\dot{\alpha}|^2 - \pi\rho\mathbf{u}^2\left(\frac{1}{2}+\mathbf{a}\right)|\alpha|^2 +$$

$$\widetilde{\mathbf{S}}\left[\dot{\alpha}\dot{\overline{\mathbf{h}}} + \dot{\overline{\alpha}}\dot{\mathbf{h}}\right] + 2\pi\rho\mathbf{u}\int_{0}^{t}|\dot{\mathbf{h}} + \left(\frac{1}{2}-\mathbf{a}\right)\dot{\alpha} + \frac{\mathbf{u}}{2}\alpha|^2 d\tau dx$$

Here,

- $\frac{1}{2}\int_{-L}^{0} E|h''(x,t)|^2 dx$ - **Potential energy due to bending displacement**
- $\frac{1}{2}\int_{-L}^{0} \widetilde{m}|\dot{h}(x,t)|^2 dx$ - **Kinetic energy due to bending displacement**
- $\frac{1}{2}\int_{-L}^{0} G|\alpha'(x,t)|^2 dx$ - **Potential energy due to torsional displacement**

$$\mathbf{E(t)} = \frac{\mathbf{1}}{\mathbf{2}}\int_{-\mathbf{L}}^{\mathbf{0}}\{\mathbf{E}|\mathbf{h}''|^{\mathbf{2}} + \mathbf{G}|\alpha'|^{\mathbf{2}} + \widetilde{\mathbf{m}}|\dot{\mathbf{h}}|^{\mathbf{2}} + \widetilde{\mathbf{I}}|\dot{\alpha}|^{\mathbf{2}} - \pi\rho\mathbf{u}^{\mathbf{2}}(\frac{\mathbf{1}}{\mathbf{2}} + \mathbf{a})|\alpha|^{\mathbf{2}} + \widetilde{\mathbf{S}}[\dot{\alpha}\dot{\overline{\mathbf{h}}} + \dot{\overline{\alpha}}\dot{\mathbf{h}}] + \mathbf{2}\pi\rho\mathbf{u}\int_{\mathbf{0}}^{\mathbf{t}}|\dot{\mathbf{h}} + \left(\frac{\mathbf{1}}{\mathbf{2}} - \mathbf{a}\right)\dot{\alpha} + \frac{\mathbf{u}}{\mathbf{2}}\alpha|^{\mathbf{2}}\}\mathbf{d}\tau\}\mathbf{dx}$$

Here,

- $\frac{1}{2}\int_{-L}^{0}\widetilde{I}|\dot{\alpha}(x,t)|^2dx$ - **Kinetic energy due to torsional displacement**
- $\frac{1}{2}\int_{-L}^{0}\widetilde{S}[\dot{\alpha}\dot{\overline{h}} + \dot{\overline{\alpha}}\dot{h}](x,t)dx$ - **Kinetic energy due to bending-torsion coupling**
- $\pi\rho u\int_{-L}^{0}dx\int_{0}^{t}|\left(\dot{h} + \left(\frac{1}{2} - a\right)\dot{\alpha} + \frac{u}{2}\alpha\right)(x,\tau)|^2d\tau$ - **Energy of vibration due to air-structure interaction**

Theorem. For each value of the airspeed u, there exists $\mathcal{R}(u) > 0$ such that the following statement holds. If an eigenvalue λ_n satisfies $|\lambda_n| > \mathcal{R}(u)$, then this eigenmode is stable, i. e., $\Re\lambda_n < 0$. The following estimate is valid for $\mathcal{R}(u)$:

$$\mathcal{R}(u) = \mathcal{C}\sqrt{\frac{\rho}{G\ \ \Re\ \ \delta}}\ u^{3/2}$$

with $\mathcal{C}$ being an absolute constant.

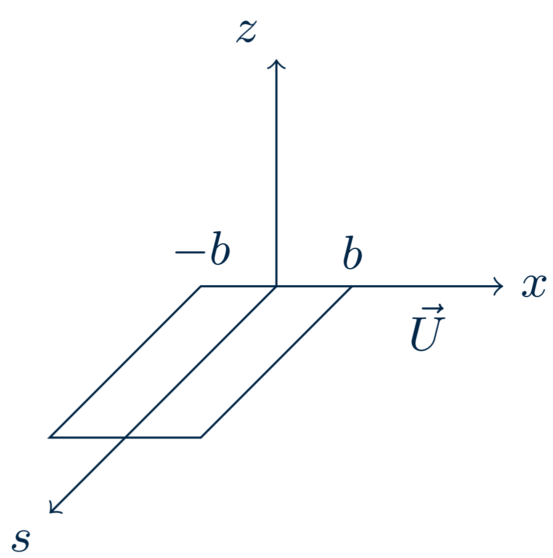
Corollary. For each u, all unstable modes are located inside the "circle of instability" $|\lambda| = \mathcal{R}(u)$. The number of these eigenmodes is finite.

Possio Integral Equation in Aeroelasticity

Physical assumptions on airflow

1. **Flow $\Rightarrow$ nonviscous fluid $\Rightarrow$ Euler equation for stream velocity $\vec{v}$**
2. **Compressible Flow:** $\rho_t + \nabla \bullet (\rho \vec{v}) = 0$ **(continuity equation)**
3. **Isentropic flow:** $P = k\rho^{\gamma}$
4. **Irrotational (or potential) flow:** $\vec{v} = \nabla \Phi$
5. **Subsonic** $(0 < M < 1)$
6. **Coupling between structure and airflow**
   - **Flow Tangency Condition**
     (Flow is attached to wing surface)
   - **Kutta-Joukowsky Condition**
     (Pressure drop off the wing and on trailing edge)

The Possio Integral Equation relates pressure distribution over a typical section of a slender wing to a normal velocity of the points on a wing surface ("downwash")



Model: High aspect-ratio planar wing; all cross-sections along wing-span are identical; only one spatial variable along cord

- Velocity field → potential U-free stream velocity
  Velocity potential:

$$U + \phi(x, z, t),$$
$$-\infty < x < \infty, \quad 0 < z < \infty,$$
$$t > 0$$

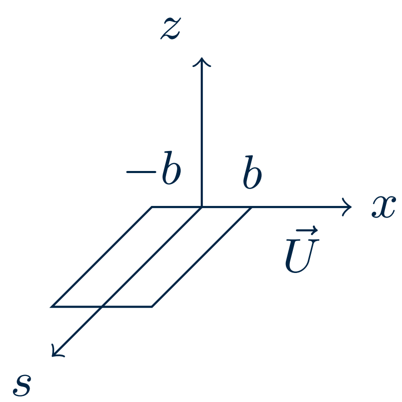
Linearized version of Euler Equation for disturbance potential

$$\frac{\partial^2\phi}{\partial^2 t} + 2Ma_\infty \frac{\partial^2\phi}{\partial t\partial x} = a_\infty^2(1-M^2)\frac{\partial^2\phi}{\partial^2 x} + a_\infty^2\frac{\partial^2\phi}{\partial^2 z}$$

a_∞-speed of sound in flight altitude
$M = U/a_\infty$-Mach number

Boundary conditions make the problem complicated

- **Flow Tangency Condition (flow is attached)**
- **Kutta-Joukowsky Condition (zero pressure off the wing and at the trailing edge)**
- **Far Field conditions**



1. **Flow Tangency Condition**

$$\frac{\partial}{\partial z}\phi(x,z,t)|_{z=0} = w_a(x,t), \quad |x| < b$$

w_a-**given normal velocity of wing**

2. **Kutta-Joukowsky Condition**
Acceleration potential:

$$\psi(x,z,t) = \frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x}$$

- $\psi(x,0,t) = 0, \quad |x| > b$
- $\lim_{x \to b-0} \psi(x,0,t) = 0$

3. **Far Field conditions**
Disturbance potential $\to$ **0 as** $|x| \to \infty$ **or** $z \to \infty$

The Possio Integral Equation

$$
\begin{aligned}
W_a(\cdot,\lambda) \quad = \quad & \frac{\sqrt{1-M^2}}{2}\Big\{\mathcal{H}_b A(\cdot,\lambda) - \frac{1}{\sqrt{1-M^2}} \times \\
& \left[\tilde{\lambda}\mathcal{H}_b\mathcal{L}(\tilde{\lambda})A(\cdot,\lambda) - \tilde{\lambda}g_-(\tilde{\lambda},x)e^{-b\tilde{\lambda}}L\left(\tilde{\lambda},A(\cdot,\lambda)\right)\right] - \\
& \int_0^{\alpha_1} a(s)\left[\tilde{\lambda}\mathcal{H}_b\mathcal{L}(\tilde{\lambda}s)A(\cdot,\lambda) - \right. \\
& \qquad \left. \tilde{\lambda}g_-(\tilde{\lambda}s,x)e^{-\tilde{\lambda}bs}L\left(\tilde{\lambda}s,A(\cdot,\lambda)\right)\right]ds + \\
& \int_0^{\alpha_2} a(-s)\left[\tilde{\lambda}\mathcal{H}_b\mathcal{L}^*(\tilde{\lambda}s)A(\cdot,\lambda) + \right. \\
& \qquad \left. \tilde{\lambda}g_+(\tilde{\lambda}s,x)e^{-\tilde{\lambda}bs}L\left(-\tilde{\lambda}s,A(\cdot,\lambda)\right)\right]ds\Big\}
\end{aligned}
$$

Asymptotical form of Possio Equation as $\lambda \to \infty$

$$
\begin{aligned}
2\mathcal{T}W_a(\cdot,\lambda) \quad &= \quad \left[\sqrt{1-M^2} - \tilde{\lambda}\mathcal{L}(\tilde{\lambda})\right] F(\cdot,\lambda) + \\
&\tilde{\lambda} e^{-b\tilde{\lambda}} h_-(x,\tilde{\lambda}) L(\tilde{\lambda}, F(\cdot,\lambda)) - \\
&M\left\{\mathcal{H}_b[F(\cdot,\lambda)] - \mathcal{T}[F(\cdot,\lambda)]\right\}
\end{aligned}
$$

Main difficulty:

$\mathcal{L}(\lambda) \quad \to$ **Volterra integral operator**

$L(\tilde{\lambda},\cdot) \quad \to$ **Integral operator with degenerate kernel**

$\mathcal{H}_b(\cdot) \quad \to$ **Finite Hilbert transform**

$\mathcal{T}(\cdot) \quad \to$ **"Inverse" to $\mathcal{H}_b$**

$\mathcal{H}_b$ **and** $\mathcal{T} \to$ **Singular integral operators**