

PROBLEM SESSION
50 YEARS OF NUMBER THEORY AND RANDOM MATRIX
THEORY

- (1) Michael Rubinstein. The following random matrix integral arose in the work [?] in connection with the k -fold divisor function in short intervals:

$$\int_{U(N)} \det(I - xU)^k \det(I - \bar{U})^k dU = \sum_{m=0}^{kN} I_k(m, N) x^m.$$

My former Master's student Andy showed in Theorem 4 of his thesis that this is equal to

$$\frac{c_{N,k}}{(1-x)^{k^2}} \underbrace{\det \left(\frac{1-x^{N+i+j-1}}{N+i+j-1} \right)_{i,j=1}^k}_{=: G_{k,N}(x)}$$

with

$$c_{N,k} = \prod_{j=1}^k \frac{(N+k-j-1)!}{(j-1)!^2 (N+j-1)!}.$$

I have observed, experimentally, that the function $F := xG'/G$ seems to satisfy the following differential equation:

$$\begin{aligned} x^2(x-1)^2 F''' + x(5x-1)(x-1)F'' + 6x(x-1)^2(F')^2 + 4(x-1)(x+1)FF' \\ + ((-4k^2 - 4Nk - N^2 + 4)x^2 + (4k^2 + 4Nk + 2N^2 - 2)x - N^2)F' \\ + 2F^2 + (-2k^2 - 2Nk)F = 0 \end{aligned}$$

I would like a proof that F does indeed satisfy the above equation, and a point of view that explains why.

- (2) Jared Lichtman. Show that the following sequence of integrals converges to $e^{-\gamma}$:

$$\int_0^1 \frac{dx}{1+x}, \quad \int_{[0,1]^2} \frac{dx dy}{1+x(1+y)}, \quad \int_{[0,1]^3} \frac{dx dy dz}{1+x(1+y(1+z))}, \dots$$

This arose while thinking about numbers with k prime factors, $k \rightarrow \infty$. This is related to the de Bruijn/Dickman function. Hoping for some nice direct proof, which would then generalize to related sequences of integrals. So far, only have a circuitous route to showing this using results on the distribution of numbers with k prime factors. Checks out numerically.

- (3) Matthew Young. Come up with a recipe for conjecturing the size of the norm in the large sieve inequality for a family of automorphic forms:

$$\Delta(\mathcal{F}, N) = \max_{\|\alpha\|=1} \sum_{f \in \mathcal{F}} \left| \sum_{n \leq N} a_n \lambda_f(n) \right|^2.$$

Here \mathcal{F} is a “primitive” family of automorphic forms. Avoid “biased sets.” Optimistic bound is $N + |\mathcal{F}|$, maybe up to $(N|\mathcal{F}|)^\varepsilon$. Maybe don’t get that in general, but would be nice at least to have a recipe saying how big this thing is.

- Henryk Iwaniec: A comment. There are conjectures of this type for Dirichlet characters where the families are not complete. Work of Bombieri, Montgomery. Works of Heath-Brown [?] and Dunn-Radziwiłł [?] on Patterson’s conjecture.
 - Maybe nice to have more examples?
- (4) Brad Rodgers. A problem more in random matrix theory and analysis than number theory. Let ω be uniformly distributed on the unit circle S^1 . Define a 2×2 matrix $g(\omega)$ by

$$g(\omega) := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \omega \\ 1 & -\omega \end{pmatrix} \in U(2).$$

Consider a sequence of integers n_k which is lacunary in the sense that $n_{k+1}/n_k \geq \lambda > 1$. Consider the sequence of matrix products

$$g(\omega^{n_k})g(\omega^{n_{k-1}})\cdots g(\omega^{n_1}).$$

The conjecture is that this random variable tends to the Haar measure on $U(2)$. Currently known in only two special cases, but not in general:

- For $n_k = \lambda^k$ via Brad’s work on Rudin-Shapiro polynomials [?]
 - For $n_{k+1}/n_k \rightarrow \infty$ (master’s thesis under preparation).
- (5) Aled Walker. Consider Montgomery’s Pair Correlation function [?]

$$F(\alpha, T) = \left(\frac{T}{2\pi} \log T \right)^{-1} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma')$$

where $w(u) = 4/(4 + u^2)$ and γ, γ' are ordinates of the zeros of ζ .

Prove that there is an explicit $\alpha > 3/2$ for which $F(\alpha, T) \geq c > 0$ for all $t \geq T_0$, with c and T_0 explicit. Feel free to assume GRH. This is motivated by problems concerning the von Mangoldt function.

A result due to Goldston, Gonek, Özlük and Snyder [?] falls just short of that, it shows that under GRH we have $F(\alpha, T) \geq 3/2 - \alpha$ for $\alpha \in [1, 3/2]$.

- (6) Jon Keating. A challenge or problem directed at those who specialize in numerical computations. Compute a negative moment of $\int_0^T |\zeta(1/2 + \alpha + it)|^{-2k} dt$ that distinguishes between the conjectures Alexandra Florea stated in her lectures this morning: one coming from RMT while the other was made by Gonek [?].
- Brian Conrey: if α gets too small, then it’s just a few individual zeros that control everything.
 - Zeev Rudnick. To investigate these negative moments in the function field setting, even in the large- q limit, one would need to think a little bit, because functions are not continuous; inverse characteristic functions have singularities on the unit circle. (Ideally, with fixed q ?).
- (7) Zeev Rudnick. There’s a lemma of Chebyshev that says that the least common multiple of the first N integers has logarithm given by the Chebyshev ψ function at N , which is asymptotic to N by the Prime Number Theorem:

$$\log \text{lcm}\{1, \dots, N\} = \psi(N) \sim N.$$

In 2011, Javier Cilleruelo raised a variant: take a fixed irreducible polynomial f with integral coefficients, and try to estimate

$$\log \operatorname{lcm}\{f(1), \dots, f(N)\}.$$

He proved that for quadratic f this is asymptotic to $N \log N$ [?]. He conjectured that this is asymptotic to $(\deg f - 1)N \log N$ for $\deg f \geq 2$. Nothing known for degrees higher than 2. Prove his conjecture in new cases.

- (8) Dorian Goldfeld. Has anyone made any conjecture on moments of the Selberg zeta function (for which the analogue of the Riemann hypothesis is known)? Say for $\operatorname{PSL}_2(\mathbb{Z}) \backslash \mathbb{H}$. Answer probably sensitive to issues of arithmeticity (Poisson vs. GOE). Maybe a central limit theorem for $\log |Z(\frac{1}{2} + it)|$.
- Zeev Rudnick: maybe fun to do the same but, rather than averaging over t , to average over the Riemann surface. How about gaps between eigenvalues on a Riemann surface. Arbitrarily small gaps? Arbitrarily long gaps? Don't know whether they exist.
 - Matthew Young: maybe don't even know that there are infinitely many simple zeros? For $\operatorname{SL}_2(\mathbb{Z})$, the multiplicity could hypothetically be huge: we only know multiplicity one, but don't know that the eigenvalues are distinct.
- (9) Sieg Baluyot. Analogue in RMT for twisting by $(n/m)^{it}$? How about for function fields?
- (10) Jeff Lagarias. One of the mysterious questions about the Riemann zeta function, as compared with the Selberg zeta function, is: why is the Riemann zeta function entire of order one, while the Selberg zeta function is of order two? Let's give a version of this problem. Consider the following Dirichlet series:

$$f_b(s) := \sum_{n=1}^{\infty} d_b(n)n^{-s}, \quad d_b(n) = \text{sum of the digits of } n \text{ in base } b.$$

If $b \geq 2$, then it is known that this function continues meromorphically to the whole plane with simple poles that are arranged in a left half-plane at elements of a two-dimensional half-lattice with periods 1 and $\frac{2\pi i}{\log b}$ (see the paper by his student Everlove [?]). Thus the poles form a two-dimensional half-lattice. Prove that this function is of order 2 (it must be ≥ 2 because it has too many poles). On the other hand, if $b = 1$, then it's a shift of ζ , hence of order 1.

- (11) Dan Goldston. Finding the most commonly occurring gap between consecutive primes up to x is sometimes called the "jumping champion problem" (Conway). Say $x = 7$. Then we have $3 - 2 = 1, 5 - 3 = 2, 7 - 5 = 2$, so the jumping champion is 2. We ask another question. Erdős–Straus [?] in 1980 proved that the jumping champions actually have to go to ∞ , but that assumes a Hardy–Littlewood prime pair conjecture. If you assume more conjectures, you can say more. Would be nice to prove *anything* without any condition, beyond trivialities like proving that 1 is not a jumping champion. Can't prove anything, period; can't even numerically discover anything. Worked on the problem of trying to find the biggest loser, i.e., some number that will never be a jumping champion, like 2 or powers of 2. Couldn't do that. Can't prove that any given number will never be a

jumping champion. Maybe can prove that there *exists* some even number that is not eventually a jumping champion (i.e., a “loser”)? Don’t need to specify it; just show that it exists.

Problem: literally prove anything.

(See works with Ledoan [?, ?].)